Final Exam

Linear Algebra: Matrix Methods

MATH 2130

Fall 2025

Tuesday December 9, 2025

UPLOAD THIS COVER SHEET!

NAME: ____

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points:	20	20	20	20	10	10	100
Score:							

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 70 minutes to complete the exam.

1. (20 points) • Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Use the Gram–Schmidt process to find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$.

SOLUTION:

Solution. An orthonormal basis is given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

We start by finding an orthogonal basis. We have

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ -1/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

For simplicity, we will take

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

We have

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3.\mathbf{v}_2}{\mathbf{v}_2.\mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{15}\begin{bmatrix} 0 \\ 0 \\ 15 \\ 15 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} 2 \\ -4 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 \\ -9 \\ 9 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Again for simplicity we take

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Note that since $\mathbf{x}_4 = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3$, we see that $\mathbf{x}_4$ is in the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, so if we perform Gram–Schmidt to $\mathbf{x}_4$, we will get $\mathbf{v}_4 = 0$. I omit the computation here for brevity (but you should check!).

Therefore, an orthogonal basis for the span of $\mathbf{x}_1, \ldots, \mathbf{x}_4$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Consequently, an orthonormal basis is given by

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

□

Total for Question **1: 20**

2. (20 points) • For the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ of the previous problem, find the vector in the span of those vectors that is closest to the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

SOLUTION:

Solution. The closest vector to $\mathbf{v}$ in the span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ is the orthogonal projection of $\mathbf{v}$ onto the span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$, which is the vector

$$\boxed{\frac{1}{7}\begin{bmatrix} 5 \\ 8 \\ 6 \\ 8 \end{bmatrix}.}$$

Here is how to find the orthogonal projection. In the previous problem, we computed an orthonormal basis for the span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ to be:

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}.$$

The orthogonal projection of $\mathbf{v}$ onto the span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ is then given by

$$(\mathbf{v}.\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}.\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v}.\mathbf{u}_3)\mathbf{u}_3$$

$$= \left(\frac{1}{3}\right)(3)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{15}(3)\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} + \frac{1}{35}(-3)\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

$$= \frac{35}{35}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{7}{35}\begin{bmatrix}-1\\2\\3\\-1\end{bmatrix} - \frac{3}{35}\begin{bmatrix}1\\3\\-3\\-4\end{bmatrix} = \frac{1}{35}\begin{bmatrix}25\\40\\30\\40\end{bmatrix} = \frac{1}{7}\begin{bmatrix}5\\8\\6\\8\end{bmatrix}$$

□

Total for Question **2: 20**

3. (20 points) • *Find the equation $y = \beta_0 + \beta_1 x$ of the line that best fits the given data points, as a least squares model:*

$$\begin{bmatrix} x \\ y \end{bmatrix} : \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

SOLUTION:

Solution. The best fit line is

$$\boxed{y = \frac{4}{5} + \frac{2}{5}x}$$

To find this, we have the matrices:

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \widetilde{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$$

The best fit line is given by $\boldsymbol{\beta}$ satisfying

$$\widetilde{\mathbf{x}}^T \widetilde{\mathbf{x}} \boldsymbol{\beta} = \widetilde{\mathbf{x}}^T \mathbf{y}.$$

In other words, we are trying to solve

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

In other words,

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

□

Total for Question **3: 20**

4. • Consider the following real matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) (5 points) *Find the characteristic polynomial $p_A(t)$ of A.*

SOLUTION:

Solution to (a). The characteristic polynomial of A is:

$$\boxed{p_A(t) = \det(tI - A) = t^3 - 11t^2 + 36t - 36}$$

If you used the textbook's convention, you will get $p_A(t) = \det(A - tI) = 36 - 36t + 11t^2 - t^3$; that is also fine.

Here is the computation.

$$\det(tI - A) = \begin{vmatrix} t-3 & +1 & -1 \\ +1 & t-5 & +1 \\ -1 & +1 & t-3 \end{vmatrix}$$

$$\begin{aligned}
&= (t-3)[(t-5)(t-3) - (1)(1)] - (1)[(t-3) - (1)(-1)] + (-1)[(1)(1) - (t-5)(-1)] \\
&= (t-3)[t^2 - 8t + 15 - 1] - [t - 3 + 1] - [1 + t - 5] \\
&= (t-3)[t^2 - 8t + 14] - [t-2] - [t-4] \\
&= [t^3 - 8t^2 + 14t - 3t^2 + 24t - 42] - 2t + 6 \\
&= t^3 - 11t^2 + 36t - 36
\end{aligned}$$

□

(b) (5 points) *Find the eigenvalues of A.*

SOLUTION:

Solution to (b). The eigenvalues of A are

$$\boxed{\lambda = 6, 3, 2}$$

The computation is as follows. By trying, $0, \pm 1, \pm 2$, we see that $p_A(2) = 0$. Thus we have

$$\begin{aligned} p_A(t) &= t^3 - 11t^2 + 36t - 36 \\ &= (t-2)(t^2 - 9t + 18) \\ &= (t-2)(t-3)(t-6) \end{aligned}$$

Therefore, the real roots of $p_A(t)$ are $\lambda = 6, 3, 2$. □

(c) (5 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

SOLUTION:

Solution to (c). A basis for each eigenspace is:

$$\boxed{E_6 \leftrightarrow \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad E_3 \leftrightarrow \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad E_2 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.}$$

The computation is as follows. We start with E_6. We want to find a basis for the kernel of

$$6I - A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ 3 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want.

Next we consider E_3. We want to find a basis for the kernel of

$$3I - A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want.

Finally we consider E_2. We want to find a basis for the kernel of

$$2I - A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We then modify the matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red -1, gives the basis element we want. □

(d) (5 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

SOLUTION:

Solution to (d). Yes, A is diagonalizable. We can use the matrix with columns given by the basis elements for the eigenspaces that we just computed. In other words, we may take

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}.$$

□

Total for Question **4: 20**

5. • Consider the 2-dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

(a) (5 points) *Is the origin an attractor, repeller, or saddle point?*

SOLUTION:

Solution. The origin is a saddle point.

To see this, we compute that the characteristic polynomial is

$$p_A(t) = \det \begin{pmatrix} t-1.7 & -0.3 \\ -1.2 & t-0.8 \end{pmatrix} = (t^2 - 2.5t + 1.36) - (.36) = t^2 - 2.5t + 1$$

$$= (t-2)(t-\frac{1}{2})$$

Thus the eigenvalues are $\lambda = \frac{1}{2}, 2$. Since $0 < \frac{1}{2} < 1$ and $1 < 2$, we see that the origin is a saddle point. □

(b) (5 points) *Find the directions of greatest attraction or repulsion.*

SOLUTION:

Solution. We have that the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion.

To deduce this, we find the eigenspaces. We start with the $\lambda = \frac{1}{2}$-eigenspace, $E_{1/2}$, which is the kernel of $\frac{1}{2}I - A$:

$$\frac{1}{2}I - A = \begin{pmatrix} -1.2 & -0.3 \\ -1.2 & -0.3 \end{pmatrix} \mapsto \begin{pmatrix} 12 & 3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & -1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is a basis for the $\frac{1}{2}$-eigenspace $E_{1/2}$.

We now compute the $\lambda = 2$-eigenspace, E_2, which is the kernel of $2I - A$:

$$2I - A = \begin{pmatrix} 0.3 & -0.3 \\ -1.2 & 1.2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the 2-eigenspace E_2.

In conclusion, the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion. □

Total for Question **5: 10**

6. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do not need to justify your answer**.

(a) (2 points) **TRUE** or **FALSE** (circle one). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

SOLUTION: TRUE: This is Cauchy–Schwarz.

(b) (2 points) **TRUE** or **FALSE** (circle one). *Two vectors in* $\mathbb{R}^n$ *are orthogonal if their dot product is zero.*

SOLUTION: TRUE: This was our definition of orthogonal.

(c) (2 points) **TRUE** or **FALSE** (circle one). *If* $A \in M_{m\times n}(\mathbb{R})$ *and* $\mathbf{b} \in \mathbb{R}^m$, *then a least squares solution to the equation* $A\mathbf{x} = \mathbf{b}$ *is a vector* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *such that* $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

SOLUTION: TRUE: We showed this in class – this is Theorem 13, p.383 of Lay.

(d) (2 points) **TRUE** or **FALSE** (circle one). *If A is any real matrix, then the matrix* A^TA *has non-negative eigenvalues.*

SOLUTION: TRUE: Considering an eigenvector $\mathbf{x}$ for A^TA with eigenvalue λ, one has $0 \leq \|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda\|\mathbf{x}\|^2$. Dividing by $\|\mathbf{x}\|^2 > 0$ gives the assertion.

(e) (2 points) **TRUE** or **FALSE** (circle one). *Given symmetric matrices A and B of the same size, i.e.,* $A = A^T$ *and* $B = B^T$, *then AB is a symmetric matrix, i.e.,* $AB = (AB)^T$.

SOLUTION: FALSE: For instance, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

Total for Question **6: 10**