Exercise 5.4.25

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 5.4.25 from Lay [LLM16, §5.4]:

Exercise 5.4.25. The *trace* of a square matrix A is the sum of the diagonal entries in A and is denoted by $\operatorname{tr} A$. It can be verified (see below) that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then $\operatorname{tr} A = \operatorname{tr} B$.

Solution. Suppose that A and B are similar. Then there exists an invertible matrix S such that $B = S^{-1}AS$. We then have

$$\operatorname{tr}(B) = \operatorname{tr}(S^{-1}AS) = \operatorname{tr}(S^{-1}(AS)) = \operatorname{tr}((AS)S^{-1}) = \operatorname{tr}(A).$$

□

Remark 0.1. We can prove $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ as follows.

$$\operatorname{tr}(FG) = \sum_{i=1}^{n}(FG)_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} F_{ik}G_{ki}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} F_{ik}G_{ki}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} G_{ki}F_{ik}\right) = \sum_{k=1}^{n}(GF)_{kk} = \operatorname{tr}(GF).$$

Remark 0.2. Another way to prove $\operatorname{tr} A = \operatorname{tr} B$ is through the characteristic polynomial. If two matrices A and B are similar, then $p_A(t) = p_B(t)$, since if $B = S^{-1}AS$, we have

$$p_A(t) = \det(tI - A) = \det(S^{-1}(tI - A)S) = \det(tI - S^{-1}AS) = \det(tI - B) = p_B(t).$$

We also know that $p_A(t) = t^n - \operatorname{tr}(A)t^{n-1} + \cdots + (-1)^n \det(A)$, so that the equality $p_A(t) = p_B(t)$ implies that $\operatorname{tr}(A) = \operatorname{tr}(B)$.

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References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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