Exercise 5.5.27

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 5.5.27 from Lay [LLM21, §5.5]:

Exercise 5.5.27. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q(\mathbf{x}) = \overline{\mathbf{x}}^T A\mathbf{x}$. Show that $q(\mathbf{x})$ is a real number.

Solution. We will show more generally that if A is an $n \times n$ *complex* matrix with the property that $\bar{A}^T = A$, and $\mathbf{x}$ is any vector in $\mathbb{C}^n$, then $q(\mathbf{x}) = \overline{\mathbf{x}}^T A\mathbf{x}$ is a real number. To do this, it suffices to show that $\overline{q(\mathbf{x})} = q(\mathbf{x})$.

Indeed, we have

$$\begin{aligned}
\overline{q(\mathbf{x})} &= \overline{\overline{\mathbf{x}}^T A\mathbf{x}} \\
&= \mathbf{x}^T \overline{A\mathbf{x}} && \text{(see p.307, } \overline{BC} = \overline{B}\,\overline{C}\text{, and } \overline{\overline{B}} = B\text{)} \\
&= \mathbf{x}^T \bar{A}\overline{\mathbf{x}} && (\overline{BC} = \overline{B}\,\overline{C}) \\
&= (\mathbf{x}^T \bar{A}\overline{\mathbf{x}})^T && \text{(transpose of a } 1 \times 1 \text{ matrix)} \\
&= \overline{\mathbf{x}}^T \bar{A}^T (\mathbf{x}^T)^T && \text{(see Thm. 3, p.105, } (BC)^T = C^T B^T\text{)} \\
&= \overline{\mathbf{x}}^T A\mathbf{x} && ((B^T)^T = B\text{, and we assumed } \bar{A}^T = A\text{)} \\
&= q(\mathbf{x}).
\end{aligned}$$

□

Date: November 16, 2025.

REFERENCES

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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