Exercise 4.6.6

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 4.6.6 from Lay [LLM21, §4.6]:

Exercise 4.6.6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose that

$$\begin{array}{rcrcrcr} \mathbf{f}_1 &=& 2\mathbf{d}_1 &-& \mathbf{d}_2 &+& \mathbf{d}_3 \\ \mathbf{f}_2 &=& 0\mathbf{d}_1 &+& 3\mathbf{d}_2 &+& \mathbf{d}_3 \\ \mathbf{f}_3 &=& -3\mathbf{d}_1 &+& 0\mathbf{d}_2 &+& 2\mathbf{d}_3 \end{array} \tag{0.1}$$

a. Find the change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ to the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$.

b. Find the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ for the vector

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3.$$

Solution. a. The change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ to the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ can be read off from (0.1) as the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ -3 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

b. The coordinates for the vector

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$$

with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ are given by

$$(1, -2, 2).$$

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Therefore, given part a., the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are given by

$$\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

In other words, the coordinates for $\mathbf{x}$ with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are

$$(-4, -7, 3).$$

□

Remark 0.1. Note that in our solution to b., we are claiming that

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3 = -4\mathbf{d}_1 - 7\mathbf{d}_2 + 3\mathbf{d}_3.$$

We could have checked this directly by substituting in the following way:

$$\begin{aligned} \mathbf{x} &= \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3 \\ &= (2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3) - 2(3\mathbf{d}_2 + \mathbf{d}_3) + 2(-3\mathbf{d}_1 + 2\mathbf{d}_3) \\ &= -4\mathbf{d}_1 - 7\mathbf{d}_2 + 3\mathbf{d}_3 \end{aligned}$$

REFERENCES

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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