Exercise 4.2.26

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 4.2.26 from Lay [LLM16, §4.2]:

Exercise 4.2.26. True or False. Justify your answer.

A kernel ("null space") is a vector space.

Solution. TRUE. Given a linear map $L : V \to V'$ of vector spaces, we have seen that the kernel of L is a sub-space of V, and therefore is a vector space (see also [LLM16, Theorem 2, p.212]). For convenience, here is the proof. To show that $\ker(L)$ is a subspace of V, we must show that $\ker(L)$ is closed under addition and scaling. In other words, for all $v, w \in \ker(L)$, and all $\lambda \in \mathbb{R}$, we must show

(1) $v + w \in \ker(L)$

(2) $\lambda v \in \ker(L)$.

Let us first show (1). We are assuming that $v, w \in \ker(L)$ (recall that $\ker(L) = \{x \in V : L(x) = 0\}$), and we must show that $v + w \in \ker(L)$. We have $L(v + w) = L(v) + L(w) = 0 + 0 = 0$, so that $v + w \in \ker(L)$. For (2), we have $\lambda v \in \ker(L)$, since $L(\lambda v) = \lambda L(v) = \lambda \cdot 0 = 0$. □

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REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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