Exercise 2.9.9

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 2.9.9 from Lay [LLM21, §2.9]:

Exercise 2.9.9. In this problem, we display a matrix A and a row echelon form of A. Find bases for the column space, $\mathrm{Col}(A)$, and the kernel, $\ker(A)$, ("null space, $\mathrm{Nul}(A)$,"), and then state the dimensions of these subspaces.

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. A basis for the column space of A is given by the columns of A corresponding to the columns of the row echelon form with pivots. In other words, a basis for the column space of A is given by the vectors

$$\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$$

Since there are 3 basis vectors for $\mathrm{Col}(A)$, we see that $\dim \mathrm{Col}(A) = 3$.

The kernel of A is the same as the space of solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$. We can see that the RREF of A is

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Date: October 5, 2025.

Therefore, the modified matrix is

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so a basis for $\ker(A)$ is given by the vector

$$\begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

Since there is 1 basis vector for $\ker(A)$, we see that $\dim \ker(A) = 1$. $\square$

References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu