Exercise 2.7.7

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 2.7.7 from Lay [LLM21, §2.7]:

Exercise 2.7.7. Find the 3×3 matrix that produces the following composite $2D$ map ("transformation"), using homogeneous coordinates: *Rotate points through* $60°$ *(counter clock-wise) about the point* $(6, 8)$.

Solution. Let's use the notation that $T_{(a,b)}$ is the (non-linear) map that translates points in the plane by the point (a, b), and that R_θ is the linear map that rotates points in the plane (counter clock-wise) about the origin. Then the map we want is the composition $T_{(6,8)} \circ R_{60°} \circ T_{(-6,-8)}$. In other words, translate first so that $(6, 8)$ goes back to the origin, then rotate the points in the plane around the origin, and then translate back to by $(6, 8)$.

To do this with matrices, first recall that since $T_{(a,b)}$, is given by $T_{(a,b)}(x_1, x_2) = (x_1 + a, x_2 + b)$, this is given in $\mathbb{R}^3$ with third coordinate 1 by $(x_1, x_2, 1) \mapsto (x_1 + a, x_2 + b, 1)$, which in homogeneous coordinates means

$$(x_1, x_2, x_3) \mapsto (x_1 + ax_3, x_2 + bx_3, x_3).$$

(Taking $x_3 = 1$ is the map we described in $\mathbb{R}^3$.) From the above, we can see that $T_{(a,b)}$ in homogeneous coordinates is given by the linear map associated with the matrix

$$T_{(a,b)} : \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Date: October 5, 2025.

Similarly, one has that rotation through the angle θ around the origin is given by

$$R_\theta : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the composition $T_{(6,8)} \circ R_{60^\circ} \circ T_{(-6,-8)}$ corresponds to

$$T_{(6,8)} \circ R_{60^\circ} \circ T_{(-6,-8)} : \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 3+4\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$\square$

References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu