Final Exam

Intro to Discrete Math

MATH 2001

Fall 2024

Sunday December 15, 2024

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points:	20	20	20	20	20	20	120
Score:							

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 60 minutes to complete the exam.

1. (20 points) • The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by the rule that $F_1 = 1$, $F_2 = 1$, and for $n \geq 3$, one sets $F_n = F_{n-1} + F_{n-2}$. In other words, the Fibonacci sequence begins:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Give a *proof by induction* that for each natural number n the following statement is true:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

SOLUTION:

Solution. For each natural number n we have the statement:

$$p(n): \quad \sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

We start by proving $p(1) : \sum_{i=1}^{1} F_i^2 = F_1 F_{1+1}$. In other words, we must prove that $F_1^2 = F_1 F_2$. Writing this out we have $F_1^2 = 1^2 = 1 \cdot 1 = F_1 F_2$, confirming that $p(1)$ is true. Now we assume that for some $n \geq 1$ and all $m \leq n$ we have proven that $p(m)$ is true. We then must prove that the statement $p(n+1)$ is true:

$$\sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{(n+1)+1} = F_{n+1} F_{n+2}.$$

To establish this we make the following computation

$$\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2},$$

where, for the second equality, we are using the inductive hypothesis that $p(n)$ is true. This completes the proof. □

Total for Question **1: 20**

2. (20 points) • **TRUE** or **FALSE**:

If R and S are equivalence relations on a set A, then $R \subseteq S$ if and only if for all $X \in A/R$ there exists $Y \in A/S$ with $X \subseteq Y$.

If true, give a *proof* of the statement. If false, provide a *counter example*, and prove that it is a counter example. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

Recall that A/R is the set of equivalence classes for the equivalence relation R, and A/S is the set of equivalence classes for the equivalence relation S.

SOLUTION:

Solution. This statement is TRUE.

First assume that $R \subseteq S$, and let $X \in A/R$. We want to show that there exists $Y \in A/S$ such that $X \subseteq Y$. As $X \in A/R$, by definition we have that there exists $a \in A$ such that X is the equivalence class of a for R; i.e., $X = [a]_R = \{x \in A : x \sim_R a\} = \{x \in A : (x,a) \in R\}$. We can then see that

$$X = [a]_R = \{x \in A : (x,a) \in R\} \subseteq \{x \in A : (x,a) \in S\} = [a]_S \in A/S,$$

using that $R \subseteq S$. Taking the equivalence class $Y = [a]_S \in A/S$, this completes the proof that there exists $Y \in A/S$ with $X \subseteq Y$.

Conversely, assume that for all $X \in A/R$ there exists $Y \in A/S$ with $X \subseteq Y$. We want to show that $R \subseteq S$. So let $(a,b) \in R$. We have the equivalence class $[a]_R \in A/R$, and we have by assumption that there exists $Y \in A/S$ such that $[a]_R \subseteq Y$. Since we are assuming that $a \sim_R b$ (i.e., $(a,b) \in R$), we have $a, b \in [a]_R \subseteq Y$, so that $a, b \in Y$. By definition, there exists $c \in A$ such that $[Y] = [c]_S = \{x \in A : x \sim_S c\}$. Thus we have $a \sim_S c$ and $b \sim_S c$, so that $a \sim_S b$, by symmetry and transitivity. In other words, $(a,b) \in S$, completing the proof. □

Total for Question **2: 20**

3. • Answer the following questions about maps of sets.

(a) (4 points) *Write down all the maps (functions) of sets* $f : \{1,2\} \to \{1,2\}$ *by listing the values of* $f(1)$ *and* $f(2)$.

1. $f(1) =$ $\quad f(2) =$ $\qquad$ 2. $f(1) =$ $\quad f(2) =$ $\qquad$ 3. $f(1) =$ $\quad f(2) =$ $\qquad$ 4. $f(1) =$ $\quad f(2) =$

SOLUTION:

Solution. 1. $f(1) = 1 \quad f(2) = 1 \qquad$ 2. $f(1) = 1 \quad f(2) = 2 \qquad$ 3. $f(1) = 2 \quad f(2) = 1$
4. $f(1) = 2 \quad f(2) = 2$ $\square$

(b) (2 points) *Circle the maps above that are injective.*

SOLUTION:

Solution. The maps 2. and 3. above are injective. $\square$

(c) (2 points) *Are there any maps above that are injective but not surjective?*

SOLUTION:

Solution. No, all the injective maps are surjective. $\square$

(d) (5 points) *How many injective maps of sets* $f : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ *are there?* Explain.

SOLUTION:

Solution. As we saw in class, a map of sets $f : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is the same as an ordered list $a_1, a_2, \dots, a_n$ of n elements (with repitition allowed) taken from the set $\{1,\dots,n\}$; i.e., we have $f(k) = a_k$ for $k \in \{1,\dots,n\}$. For an injective map, we cannot have repetition, so there are n options for a_1, $n-1$ options for a_2, and so on. This gives $n!$ injective maps. $\square$

(e) (2 points) *How many bijective maps of sets $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there?* Explain.

SOLUTION:

Solution. There are $n!$ bijective maps of sets $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, as every such injective map is surjective. Indeed, given a map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ determined by an ordered list $a_1, \ldots, a_n$ (with repetition allowed), the image of f is the set $\{a_1, \ldots, a_n\}$. If f is injective, i.e., the list is without repetition, then the image $\{a_1, \ldots, a_n\} \subseteq \{1, \ldots, n\}$ must have n elements, and so the image is equal to $\{1, \ldots, n\}$, implying that f is surjective. □

(f) (5 points) *If A and B are finite sets, how many bijective maps of sets $f : A \to B$ are there?* Explain.

SOLUTION:

Solution. As we saw in class, if $|A| > |B|$, then there are no injective maps $f : A \to B$, by the Division Principle. Indeed, if $|A| = n > m = |B|$, and a map $f : A \to B$ is given by the ordered list $a_1, \ldots, a_n$ of n elements of B, then, by the Division Principle, there must be some $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $a_i = a_j$. In other words, we must have $f(i) = f(j)$ for some $i \neq j$. Thus, if $|A| > |B|$, there are no bijective maps $f : A \to B$.

If $|A| < |B|$, then there are no surjective maps $f : A \to B$. Indeed, if $|A| = n < m = |B|$, and $f : A \to B$ is given by the ordered list $a_1, \ldots, a_n$ of n elements of B (with repetition allowed), then the image of f is $\{a_1, \ldots, a_n\} \subseteq B$. The set $\{a_1, \ldots, a_n\}$ has at most n elements, while B has $m > n$ elements. Therefore the image of f cannot be equal to B, so that f is not surjective. One could also have concluded using the Pigeonhole Principle. Thus, if $|A| < |B|$, there are no bijective maps $f : A \to B$.

If $|A| = |B| = n$, then there are $n!$ bijective maps $f : A \to B$, using the argument in the previous part of this problem. □

Total for Question **3: 20**

4. • Consider the set $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 4x - y = -4\}$, and the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = x\sin(x)$.

(a) (5 points) *Show that Γ defines a map (function) $f : \mathbb{R} \to \mathbb{R}$?*

SOLUTION:

Solution. We have $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 4x - y\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 4x + 4\}$. Therefore, if $x \in \mathbb{R}$, then $(x, x^2 + 4x + 4) \in \Gamma$. In addition, if $(x, y_1), (x, y_2) \in \Gamma$, then $y_1 = x^2 + 4x + 4 = y_2$. Thus Γ defines a function $f : \mathbb{R} \to \mathbb{R}$. □

(b) (1 point) *What is $f(1)$?*

SOLUTION:

Solution. We have $(1, 1^2 + 4 \cdot 1 + 4) \in \Gamma$, so that $f(1) = 9$. □

(c) (1 point) *Write a formula for $f(x)$.*

SOLUTION:

Solution. We have $f(x) = x^2 + 4x + 4 = (x + 2)^2$. □

(d) (1 point) *What is the source (domain) of f?*

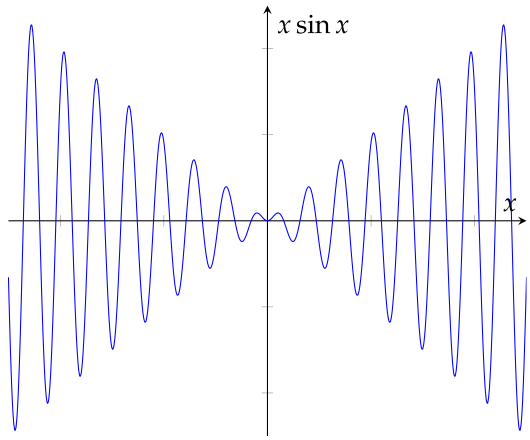
SOLUTION:

Solution. The source of f is $\mathbb{R}$. □

(e) (3 points) *What is the image of g?*

SOLUTION:

Solution. The image of g is $\mathbb{R}$. This is clear from the graph of $g(x) = x\sin x$. For this short problem, drawing the graph would be sufficient explanation:



$\square$

Here is how to show carefully that $g(\mathbb{R}) = \mathbb{R}$. First we have that $g(0) = 0$. Then, given any $y \in \mathbb{R}$, suppose first that y is positive. Then let n be any natural number such that $\pi/2 + 2\pi n > y$. For brevity, set $r = \pi/2 + 2\pi n$. We have $g(r) = r \sin(r) = r > y > 0$. Since g is differentiable (it is the product of differentiable functions) it is continuous. Therefore, by the Intermediate Value Theorem, there is some number a in the interval $[0, r]$ such that $g(a) = y$. On the other hand, if y is negative, let n be any natural number such that $-\pi/2 + 2\pi n > -y$. For brevity, set $r = -\pi/2 + 2\pi n$. We have $g(r) = r \sin(r) = -r < y < 0$. Therefore, by the Intermediate Value Theorem, there is some number a in the interval $[0, r]$ such that $g(a) = y$.

Note for later that we have actually shown the stronger statement that $g(\mathbb{R}_{\geq 0}) = \mathbb{R}$, where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.

(f) (2 points) *Write formulas for $g \circ f$ and $f \circ g$.*

SOLUTION:

Solution. We have $f(x) = (x+2)^2$ and $g(x) = x \sin x$, so that

$$(f \circ g)(x) = f(g(x)) = ((x \sin x) + 2)^2, \quad \text{and} \quad (g \circ f)(x) = g(f(x)) = (x+2)^2 \sin((x+2)^2).$$

□

(g) (1 point) *Find $(g \circ f)(1)$.*

SOLUTION:

Solution. $(g \circ f)(1) = (1+2)^2 \sin(1+2)^2 = 9 \sin 9$. □

(h) (2 points) *Find $(f \circ g)^{-1}(\{4\}) \cap \{x \in \mathbb{R} : g(x) \geq 0\}$.*

SOLUTION:

Solution. $(f \circ g)^{-1}(\{4\}) \cap \{x \in \mathbb{R} : g(x) \geq 0\} = \{n\pi : n \in \mathbb{Z}\}$. To see this, recall that we have $(f \circ g)(x) = f(g(x)) = ((x \sin x) + 2)^2$, so that

$$(f \circ g)^{-1}(\{4\}) = \{x \in \mathbb{R} : (f \circ g)(x) = 4\} = \{x \in \mathbb{R} : ((x \sin x) + 2)^2 = 4\}$$

$$= \{x \in \mathbb{R} : (x \sin x) + 2 = \pm 2\} = \{x \in \mathbb{R} : x \sin x = -2 \pm 2\}$$

$$= \{x \in \mathbb{R} : x \sin x = 0\} \cup \{x \in \mathbb{R} : x \sin x = -4\}$$

Therefore,

$$(f \circ g)^{-1}(\{4\}) \cap \{x \in \mathbb{R} : g(x) = x \sin x \geq 0\} = \{x \in \mathbb{R} : x \sin x = 0\} = \{n\pi : n \in \mathbb{Z}\}.$$

□

(i) (2 points) *Is $f \circ g$ surjective?*

SOLUTION:

Solution. No, $f \circ g$ is not surjective, as for all $x \in \mathbb{R}$, the value $(f \circ g)(x) = ((x \sin x) + 2)^2$, being the square of a real number, is never equal to -1 (or to any negative number). I.e., there is no $x \in \mathbb{R}$ such that $(f \circ g)(x) = -1$. $\square$

(j) (2 points) *Is $g \circ f$ surjective?*

SOLUTION:

Solution. Yes, $g \circ f$ is surjective, since $f(\mathbb{R}) = \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, and then we have $(g \circ f)(\mathbb{R}) = g(f(\mathbb{R})) = g(\mathbb{R}_{\geq 0}) = \mathbb{R}$, using the problem above about the image of g. $\square$

One can also do this working with individual elements as follows. Given any $y \in \mathbb{R}$, we have seen above that there exists $a \in \mathbb{R}_{\geq 0}$ such that $g(a) = y$. Then, one can see that $f(\sqrt{a} - 2) = ((\sqrt{a} - 2) + 2)^2 = a$. Consequently, $(g \circ f)(\sqrt{a} - 2) = g(f(\sqrt{a} - 2)) = g(a) = y$. Therefore, for all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $(g \circ f)(x) = y$.

Total for Question **4: 20**

5. (20 points) • Suppose that $\phi : A \to B$ and $\psi : B \to C$ are maps of sets. **TRUE** or **FALSE**:

If $\psi \circ \phi$ is surjective, then ψ is surjective.

If true, give a *proof* of the statement. If false, provide a *counter example*, and prove that it is a counter example. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

SOLUTION:

Solution. This statement is TRUE. We will prove the contrapositive, namely that *if ψ is not surjective, then $\psi \circ \phi$ is not surjective*. So, let us assume that ψ is not surjective. Then, by definition, there is some $c \in C$ such that for all $b \in B$, we have $\psi(b) \neq c$. Then for all $a \in A$, we have $(\psi \circ \phi)(a) = \psi(\phi(a)) \neq c$, since $\phi(a) \in B$. Therefore $\psi \circ \phi$ is not surjective. □

RUBRIC: See pdf.

Total for Question **5: 20**

6. (20 points) • The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (3x + 5y, x + 2y)$ is bijective. *Find its inverse.* You must show that your inverse is an inverse for f.

SOLUTION:

Solution. The inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f^{-1}(a, b) = (2a - 5b, -a + 3b)$.

To find the inverse, suppose that $f(x, y) = (a, b)$. Then we want to find x and y in terms of a and b. Using the formula for $f(x, y)$, we see that $(3x + 5y, x + 2y) = (a, b)$. This gives us two linear equations:

$$\begin{array}{rcrcl} 3x & + & 5y & = & a \\ x & + & 2y & = & b \end{array}$$

You can solve this in any way you like. Here is one approach. We can reorder the equations:

$$\begin{array}{rcrcl} x & + & 2y & = & b \\ 3x & + & 5y & = & a \end{array}$$

Then we can take -3 times the first equation and add it to the second to get the equations

$$\begin{array}{rcrcr} x & + & 2y & = & b \\ & & -y & = & a - 3b \end{array}$$

Then we can add twice the second equation to the first:

$$\begin{array}{rcrcr} x & & & = & 2a - 5b \\ & & -y & = & a - 3b \end{array}$$

This gives that $f^{-1}(a, b) = (2a - 5b, -a + 3b)$.

Let us now confirm that f^{-1} as defined above is an inverse for f. We have

$$f^{-1}(f(x, y)) = f^{-1}(3x + 5y, x + 2y) = (2(3x + 5y) - 5(x + 2y), -(3x + 5y) + 3(x + 2y)) = (x, y),$$

$$f(f^{-1}(a, b)) = f(2a - 5b, -a + 3b) = (3(2a - 5b) + 5(-a + 3b), (2a - 5b) + 2(-a + 3b)) = (a, b),$$

so that $f^{-1} \circ f = \mathrm{Id}_{\mathbb{R}^2}$ and $f \circ f^{-1} = \mathrm{Id}_{\mathbb{R}^2}$. $\square$

Total for Question **6: 20**