CHAPTER 170

Review of complex analysis in one variable

This gives a brief review of some of the basic results in complex analysis. In particular, it outlines the background in single variable complex analysis that is discussed in [Huy05, §1.1].

1. Complex numbers

We define the complex numbers $\mathbb{C}$ to be the field $(\mathbb{R}^2, +, \cdot)$ where $(\mathbb{R}^2, +)$ is the standard $\mathbb{R}$-vector space of dimension 2, and $\cdot$ is defined by $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. For convenience write $(a, b) = a + ib$. We will denote by $\mathrm{CO}(2, \mathbb{R})$ the group of real two by two conformal matrices:

$$\mathrm{CO}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : (a, b) \in \mathbb{R}^2 - \{0\} \right\}.$$

Set

$$\widehat{\mathrm{CO}}(2, \mathbb{R}) = \mathrm{CO}(2, \mathbb{R}) \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a ring under matrix addition and multiplication.

Exercise 170.1.1. *Show that there is an isomorphism of rings*

$$\phi : \mathbb{C} \to \widehat{\mathrm{CO}}(2, \mathbb{R})$$
$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Exercise 170.1.2. *Given a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a linear map $\alpha \in M(1, \mathbb{C}) = \mathbb{C}$ making the following diagram commute*

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \| & & \| \\ \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C}. \end{array}$$

if and only if $A \in \widehat{\mathrm{CO}}(2, \mathbb{R})$. In this case $A = \phi(\alpha)$. In particular, given $a + ib \in \mathbb{C}$, then multiplication of complex numbers by $a + ib$, when viewed as an $\mathbb{R}$-linear map of $\mathbb{R}^2$, is given by $\phi(a + ib)$.

2. Holomorphic maps

Definition 170.2.3 (Holomorphic map)**.** *Let $U \subseteq \mathbb{C}$ be an open subset. A map*

$$f : U \to \mathbb{C}$$

is said to be holomorphic if it at each point $p \in U$, the real differential $D_p f : \mathbb{R}^2 \to \mathbb{R}^2$ exists and is complex linear (i.e., $D_p f \in \widehat{\mathrm{CO}}(2, \mathbb{R})$).

EXAMPLE 170.2.4. A complex analytic function on an open subset of the complex plane is holomorhpic (on that open subset). We will recall below the proof that the converse holds.

EXAMPLE 170.2.5. In particular, the function $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is analytic on $\mathbb{C}$, and is therefore holomorphic.

Corollary 170.2.6. *Let $U \subseteq \mathbb{C} = \mathbb{R}^2$ be an open subset. A map $f : U \to \mathbb{C}$ that is differentiable at each point $p \in U$ is holomorphic if and only if, writing $f(x,y) = u(x,y) + iv(x,y)$, the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p), \quad \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$$

hold at each point $p \in U$.

PROOF. This follows immediately from the definitions. □

REMARK 170.2.7. The Cauchy–Riemann equations imply that if we define $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, then a differentiable function $f(z) = u(x,y) + iv(x,y)$ is holomorphic on an open set U if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$ for every $z \in U$.

Recall that if Γ is a (positively oriented) smooth contour in the complex plane, parameterized by a smooth map $\gamma : [a,b] \to \mathbb{C}$, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

EXAMPLE 170.2.8. The main example is:

$$\int_{\partial B_\epsilon(0)} z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1, \end{cases}$$

where $B_\epsilon(0)$ is the ball of radius $\epsilon > 0$ around the origin.

The main bound that one uses repeatedly is:

$$\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)|ds \leq \sup_{z\in\Gamma} |f(z)||\Gamma| \tag{88}$$

where $|\Gamma|$ is the length of the path Γ (e.g., [Rud87, §10.8 Eq. (5), p.202]). [1]

[1] [Ahl78, p.102] proof of this is as follows. For any continuous function $g : [a,b] \to \mathbb{C}$, we have

$$\left|\int_a^b g(t)dt\right| \leq \int_a^b |g(t)|\, dt.$$

Indeed, for any real θ, we have

$$\text{Re}\left[e^{-i\theta}\int_a^b g(t)\, dt\right] = \int_a^b \text{Re}\left[e^{-i\theta} g(t)\right]\, dt \leq \int_a^b |g(t)|\, dt.$$

Then taking any θ such that $\int_a^b g(t)\, dt = re^{i\theta}$, this gives the result. Finally, take $g(t) = f(\gamma(t))\gamma'(t)$ to obtain the result above.

Lemma 170.2.9. *Let $U \subseteq \mathbb{C}$ be an open subset. A continuous function $f : U \to \mathbb{C}$ is holomorphic if and only if for every $z_0 \in U$ and every open disc $B_\epsilon \subseteq U$ containing z_0 with $\overline{B}_\epsilon \subseteq U$, we have*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_\epsilon} \frac{f(z)}{z - z_0} dz. \tag{89}$$

PROOF. We sketch the proof. Suppose first that f is holomorphic. The key point is that the function $f(z)/(z - z_0)$ is holomorphic everywhere in B_ϵ except for the point z_0. Therefore, using say Stoke's Theorem, the integral in (89) is the same for every positively oriented circle $C_r := \partial B_r$ of positive radius r contained in the disk B_ϵ, and containing z_0. Now let us focus on such circles centered at z_0, and consider:

$$\begin{aligned}\int_{C_r} \frac{f(z)}{(z - z_0)} dz &= \int_{C_r} \frac{f(z_0)}{(z - z_0)} dz + \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)} dz \\ &= 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)} dz.\end{aligned}$$

Using the bound on the modulus of the integral (88), and taking the limit as r goes to 0, the integral on the right goes to 0, and one obtains (89).

For the converse, we will use a special case of [**Rud87**, Thm 10.7, p.199], and get analyticity in the process. The point is to show that

$$g(w) = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - w} dz$$

is an analytic function in w on $B_\epsilon(z_0)$. The point is that by assumption $g(w) = f(w)$. Note that holomorphicity is immediate, using that $1/z - w$ is continuously differentiable in w, to pass $\frac{\partial}{\partial \overline{w}}$ through the integral, and use that $\frac{\partial}{\partial \overline{w}} \frac{1}{z-w} = 0$ since $z \neq w$. To prove analyticity, we use that for $w \in B_\epsilon(z_0)$, we have

$$\frac{1}{z - w} = \frac{1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^n} = \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}}$$

converges uniformly in z for fixed $w \in B_\epsilon(z_0)$. Therefore, we have $\frac{f(z)}{(z-w)} = \sum_{n=0}^{\infty} f(z) \frac{(w-z_0)^n}{(z-z_0)^{n+1}}$, which, since f is continuous, can be shown to be uniformly convergent in z for fixed $w \in B_\epsilon(z_0)$. Then integrating against dz, using uniform convergence, and that $w - z_0$ is constant with respect to z, we see that $g(w) = \sum_{n=0}^{\infty} a_n (w - z_0)^n$ where $a_n = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{(z-z_0)^{n+1}} dz$. □

Corollary 170.2.10. *Let $U \subseteq \mathbb{C}$ be an open subset. Then $f : U \to \mathbb{C}$ is holomorphic if and only if it is complex analytic.*

PROOF. This is contained in the proof above. □

REMARK 170.2.11. The proof above shows that if $f : U \to \mathbb{C}$ is just assumed to be continuous, then

$$f(z_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0} dz.$$

2.1. Basic properties of holomorphic maps. Here we review a few basic facts about holomorphic maps.

2.1.1. *Local structure theorem.*

Theorem 170.2.12 (Local structure theorem)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map. Then locally, f factors as $z \mapsto z^m$, followed by a holomorphic isomorphism.*

REMARK 170.2.13. More precisely we mean that for each point $p \in U$, there is a ball $B_\epsilon(p)$, so that $f|_{B_\epsilon(p)} : B_\epsilon(p) \to f(B_\epsilon(p))$ factors as

$$B_\epsilon(p) \xrightarrow{z \mapsto (z-p)^m} B_{\epsilon^m}(0) \xrightarrow{\ g\ } f(B_\epsilon(p))$$

where g is a holomorphic isomorphism.

PROOF. To make the formulas simpler, take $p = 0$. We have immediately from analyticity that $f(z) = z^m h(z)$, where $h(z)$ is nowhere vanishing in a neighborhood of 0. We claim there is $g(z)$ such that $g(z)^m = h(z)$. In short, using that $h(z)$ is nowhere vanishing, and possibly taking a smaller neighborhood, we can define a branch of log and set $g(z) = \exp(\frac{1}{m}\log h(z))$. □

REMARK 170.2.14. To avoid technicalities with logs, just observe that $h'(z)/h(z)$ is holomorphic near 0. Therefore, using analyticity of holomorphic functions, we can find $a(z)$ such that $a'(z) = h'(z)/h(z)$. Then we have $\frac{d}{dz}(h(z)e^{-a(z)}) = 0$, so that $h(z) = Ce^{a(z)}$ for some constant C. Then we set $g(z) = e^{\frac{C}{m}a(z)}$.

REMARK 170.2.15. The number m is determined uniquely at a point $p \in U$ by the number of preimages of f near $f(p)$, or equivalently, by the order of vanishing of f at p.

Corollary 170.2.16. *The zero set of a nonconstant holomorphic function has no limit points in the domain of definition.*

PROOF. This follows immediately from the structure theorem (and the elementary case of $z \mapsto z^m$). □

2.1.2. *Open mapping theorem.*

Theorem 170.2.17 (Open mapping)**.** *Nonconstant holomorphic maps are open (take open sets to open sets).*

PROOF. This follows immediately from the local structure theorem. □

2.1.3. *Maximum principle.*

Theorem 170.2.18 (Maximum principle)**.** *Let $U \subseteq \mathbb{C}$ be open and connected. If $f : U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in U. If U is bounded and f can be extended to a continuous function $f : \overline{U} \to \mathbb{C}$, then $|f|$ takes its maximal values on the boundary ∂U.*

PROOF. Use the open mapping theorem. □

2.1.4. *Identity theorem.*

Theorem 170.2.19 (Identity theorem)**.** *If $f, g : U \to \mathbb{C}$ are two holomorphic functions on a connected open subset $U \subseteq \mathbb{C}$ such that $f(z) = g(z)$ for all z in a non-empty open subset $V \subseteq \mathbb{C}$, then $f = g$.*

REMARK 170.2.20. There are stronger versions of the identity theorem (e.g., take any subset V with limit points), but in this form it immediately generalizes to higher dimensions.

PROOF. From the corollary to the local structure theorem we have that zero sets of nonconstant holomorphic functions have no limit points (in the domain of definition). To prove the identity theorem, take the difference of the two functions and consider the zero set. □

2.1.5. *Riemann extension theorem.*

Theorem 170.2.21 (Riemann extension theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a bounded holomorphic function on a punctured disk. Then f can be extended to a holomorphic function $f : B_\epsilon(z_0) \to \mathbb{C}$.*

PROOF. The boundedness shows that $g(z) = (z - z_0)^2 f(z)$ is complex differentiable at z_0, and therefore given by a power series. Since $g(z)$ vanishes to at least order 2 at z_0, we have that $f(z)$ is analytic. □

2.1.6. *Riemann mapping theorem.*

Theorem 170.2.22 (Riemann mapping theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a simply connected open subset properly contained in $\mathbb{C}$. Then U is biholomorphic to the unit ball $B_1(0)$; i.e., there exists a bijective holomorphic map $f : U \to B_1(0)$ such that its inverse f^{-1} is also holomorphic.*

PROOF. We refer the reader to [Rud87, Thm. 14.8, p.283]. □

2.1.7. *Liouville's theorem.*

Theorem 170.2.23 (Liouville's Theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant. In particular, there is no biholomorphic map between $\mathbb{C}$ and a ball $B_\epsilon(0)$ with $\epsilon < \infty$.*

PROOF. Using analyticity, it is not hard to show that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Then on a circle C_R of radius R centered about z_0, if $|f(z)| \leq M_R$ for all $z \in C_R$, then the derivatives of f at z_0 satisfy (for each $n \in \mathbb{N}$)

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}.$$

Apply this to the first derivative, and let $R \to \infty$. □

2.1.8. *Residue theorem.*

Theorem 170.2.24 (Residue theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a holomorphic function on a punctured disk. Then f can be expanded in a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and the coefficient a_{-1} is given by the residue formula*

$$a_{-1} = \frac{1}{2\pi i} \int_{\partial B_{\epsilon/2}(z_0)} f(z) dz.$$

PROOF. The existence of the Laurent series follows from basic results on analytic functions (on annuli, we can take the sum of two analytic functions, where for one, we invert z). Then we just integrate and use uniform convergence. □

2.1.9. *Inverse function theorem.*

Theorem 170.2.25 (Inverse function)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map. If $f'(z_0) \neq 0$, then f is locally a holomorphic isomorphism near z_0.*

PROOF. Use the real inverse function theorem, and the fact that the inverse of a conformal matrix is conformal. □

REMARK 170.2.26. Using the structure theorem, one can show that if a holomorphic function is injective, then it is biholomorphic onto its image (indeed, locally it must be $z \mapsto z$, so it is locally biholomorphic; but it is a bijection so it is globally biholomorhpic). The same result will hold for maps $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$, but may fail if the dimensions of the source and target are not the same; e.g., $z \mapsto (z^3, z^2)$ is holomorphic and injective, but not biholomorphic onto its image.

2.1.10. *Schwarz lemma.*

Lemma 170.2.27 (Schwarz Lemma)**.** *Let f be a holomorphic function on an open neighborhood of the closure of the disk $B\epsilon(0)$. Assume that f vanishes to order k at the origin. If there is some real number C such that $|f(z)| \leq C$ for all $z \in \overline{B_\epsilon(0)}$, then actually there is the possibly stronger bound:*

$$|f(z)| \leq C \left(\frac{|z|}{\epsilon} \right)^k$$

for all $z \in \overline{B_\epsilon(0)}$.

REMARK 170.2.28. In short, we know the maximum, say C, of $|f(z)|$ occurs on the boundary circle $C_\epsilon(0)$. However, we can estimate how much smaller the modulus of f is on the interior, by multiplying C by the fraction of the distance we are to the boundary circle (to the power k).

PROOF. Fix $z \in B_\epsilon(0)$ and define a holomorphic function $g_z(w)$ on the open neighborhood of the closure of the disk $B\epsilon(0)$ on which f is defined, as follows: For such w, one sets

$$g_z(w) := w^{-k} f\left(w \cdot \frac{z}{|z|} \right).$$

Then for $|w| = \epsilon$ we have $|g_z(w)| \leq \epsilon^{-k} C$. The maximum principle then implies the same bound $g_z(w)| \leq \epsilon^{-k} C$ for all $|w| \leq \epsilon$. Hence

$$|z|^{-k} |f(z)| = |g_z(|z|)| \leq \epsilon^{-k} C,$$

giving the desired bound. □

3. Meromorphic functions

Let $U \subseteq \mathbb{C}$ be open. Informally, meromorphic function f on U is a ratio $f = g/h$ of holomorphic functions on U, up to the obvious equivalence. A little more precisely, it is an element of the fraction field of the integral domain of holomorphic functions on U. In higher dimensions, we will have to work a little harder, since we will not have enough global holomorphic functions to define things this way. Observing that for $U \subseteq \mathbb{C}$, multiplying by products of powers of $z - p$ for different points p (or more generally Ahlfors Theorem 7 p.195, and generalizing the proof to arbitrary open sets), one can define meromorphic functions equivalently to be functions given locally by the ratio of holomorphic functions. To make this precise, we make the following definition.

Given a nowhere dense (i.e., closure has empty interior) subset $S \subseteq U$ (e.g., S has no limit points in U), and a map of sets $f : U - S \to \mathbb{C}$, we say (S, f) is a representative for a meromorphic function on U if there exist:

- an open cover $U = \bigcup_{i \in I} U_i$,
- holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$,

satisfying

$$h_i|_{U_i - S} \cdot f|_{U_i - S} = g_i|_{U_i - S}$$

for every i. We say that $(S, f) \sim (S', f')$ if setting $S'' = S \cup S'$, then $f|_{U-S''} = f'|_{U-S''}$. A meromorphic function on U is an equivalence class of representatives.

REMARK 170.3.29. One can show that the set of meromorphic functions on U is a field.

REMARK 170.3.30. Recall that a limit point of S is a point of the closure $\overline{S}$ that is not in S. As mentioned above, if S has no limit points then it is nowhere dense. Indeed, if S has no limit points, then it is closed. If S had nonempty interior, then it would have limit points. Thus if S has no limit points, it is nowhere dense. On the other hand, the converse fails; there can be sets S that are nowhere dense that have limit points. Indeed, in $U = B_1(0)$, take $S = \{1/n : n \geq 2\}$. This set has closure with empty interior, but it has a limit point, 0.