Midterm

Abstract Algebra 1

MATH 3140

Summer 2021

Monday June 14, 2021

NAME: ____

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total
Points:	20	20	10	10	20	10	10	100
Score:								

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 90 minutes to complete the exam.

1. • Consider the following subset of real 2×2 matrices:

$$H := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\} \subseteq \mathrm{M}_2(\mathbb{R}).$$

(a) (10 points) *Show that matrix multiplication defines a binary operation on H.*

(b) (10 points) *Does the function $\phi : H \to \mathbb{R}$, given by*

$$\phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) = a,$$

give an isomorphism of the binary structure $\langle H, \cdot \rangle$ (here $\cdot$ denotes matrix multiplication) with the binary structure $\langle \mathbb{R}, + \rangle$? Explain.

1
20 points

2. (20 points) • Suppose that $\langle G, * \rangle$ is a binary structure such that:

1. The binary operation $*$ is associative.
2. There exists a **left** identity element; i.e., there exists $e \in G$ such that for all $g \in G$, we have $e * g = g$.
3. **Left** inverses exist; i.e., for all $g \in G$, there exists $g^{-1} \in G$ such that $g^{-1} * g = e$.

Show that $\langle G, * \rangle$ is a group.

2
20 points

3. (10 points) • Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $a^{-1}b \in H$. *Show that $\sim$ is an equivalence relation on G.*

3
10 points

4. (a) (5 points) • *In the group* $\mathbb{Z}_{28}$*, what is the order of the subgroup generated by the element* 18*?*

(b) (5 points) *How many generators are there for the group* $\mathbb{Z}_{28}$*?*

4
10 points

5. (a) (5 points) • *Is the permutation* $\sigma = (1,6,4)(2,5) \in S_6$ *even or odd?*

(b) (5 points) *Is the permutation* σ^2 *even or odd?*

(c) (5 points) *Compute* $|\sigma|$*; i.e., the order of* σ *in* S_6*.*

(d) (5 points) *With* σ *as above and* $\tau = (5,3,2)$*, compute* $\sigma\tau$ *(as a product of disjoint cycles).*

5
20 points

6. • Let A be a set, and let $G \leq S_A$ be a subgroup of the group of permutations S_A of A. For an element $a \in A$, define $G_a := \{\sigma \in G : \sigma(a) = a\}$.

(a) (5 points) *For $a \in A$, show that G_a is a subgroup of G.*

(b) (5 points) Let $a, b \in A$, and suppose there exists $\sigma \in G$ such that $b = \sigma(a)$. *Show that G_a and G_b have the same cardinality.*

6
10 points

7. (10 points) • Let H be a subgroup of a group G, and let $a, b \in G$.

TRUE or **FALSE**: *If* $aH = bH$, *then* $Ha^{-1} = Hb^{-1}$.

7
10 points