Take-Home Final

Abstract Algebra 1

MATH 3140

Summer 2021

Friday July 2, 2021

NAME: ____

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	Total
Points:	25	25	25	25	100
Score:					

- For the exam you may use **only the following resources** from this course: our textbook, your lecture notes, my lecture notes, your homework, the pdfs linked from the course webpage:

  `http://math.colorado.edu/~casa/teaching/21summer/3140/hw.html`

  and the quizzes and midterms we have taken on Canvas.
- You **may not use any other resources** whatsoever.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your solutions to **Canvas** as a **single .pdf** file with the questions in the correct order.
- The exam is due at 10:00 PM Friday July 2, 2021.

1. (25 points) • Let G be a group with center $Z(G)$. *Show that if $G/Z(G)$ is cyclic, then $Z(G) = G$.* [*Hint:* Show first there exists $g \in G$ such that for any $g_1 \in G$, there is a $z_1 \in Z(G)$ and $n_1 \in \mathbb{Z}$ such that $g_1 = g^{n_1} z_1$. Then show for any $g_1, g_2 \in G$ that $g_1 g_2 = g_2 g_1$.]

SOLUTION

Solution. It suffices to show that G is abelian (from the definition of the center, it follows immediately that a group G is abelian if and only if $G = Z(G)$). To show G is abelian, we must show that given $g_1, g_2 \in G$, then

$$g_1 g_2 = g_2 g_1.$$

To begin, since the group $G/Z(G)$ is cyclic, it has a generator $gZ(G) \in G/Z(G)$ for some $g \in G$. It follows that there are integers n_1, n_2 such that

$$g_1 Z(G) = (gZ(G))^{n_1} = g^{n_1} Z(G) \text{ and } g_2 Z(G) = (gZ(G))^{n_2} = g^{n_2} Z(G).$$

Equivalently, $(g^{n_1})^{-1} g_1, (g^{n_2})^{-1} g_2 \in Z(G)$. We can rewrite this by saying that there exists $z_1, z_2 \in Z(G)$ such that $(g^{n_1})^{-1} g_1 = z_1$ and $(g^{n_2})^{-1} g_2 = z_2$, or rather, $g_1 = g^{n_1} z_1$ and $g_2 = g^{n_2} z_2$. Then

$$g_1 g_2 = g^{n_1} z_1 g^{n_2} z_2 = g^{n_2} z_2 g^{n_1} z_1 = g_2 g_1$$

since by definition z_1, z_2 commute with all elements of G, and g commutes with itself. □

1
25 points

2. (a) (15 points) • In a commutative ring with unity, *show that* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

SOLUTION

Solution. Since we are in a commutative ring with unity, when writing out

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

one can deduce that the number of monomials of the form $a^k b^{n-k}$ in the expansion will be $\binom{n}{k}$, corresponding to choosing k of the n factors above from which to take an a, and then taking a b from the remaining $n-k$ factors.

Here is another argument using induction. First observe that

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} = \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!}$$

$$= \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}.$$

Now, using this, we will prove the assertion of problem using induction. We start with the case $n=1$, and we check that

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = b + a = (a+b)^1.$$

We now perform the inductive step. We assume that $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$ for all $m \leq n$ for some $n \geq 1$. We then show that

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

Here is the computation:

$$(a+b)^n(a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)(a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k}\right) + \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}\right)$$

$$= \binom{n}{0} b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) a^k b^{n+1-k} + \binom{n}{n} a^{n+1}$$

$$= \binom{n+1}{0} b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + \binom{n+1}{n+1} a^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

□

(b) (10 points) An element r of a ring R is said to be nilpotent if there exists some $n \in \mathbb{N}$ such that $r^n = 0$. Let N be the set of nilpotent elements of a commutative ring R with unity. *Show that N is an ideal in R.*

SOLUTION

Solution. First we will show that the set of nilpotents is a subgroup. Since $0 \in N$, we have that N is nonempty. Now, let $a, b \in N$, we will show that $(a - b) \in N$. To do this, suppose that $\alpha, \beta \in \mathbb{N}$ are such that $a^\alpha = b^\beta = 0$. Let n be an integer such that $n \geq \alpha + \beta$. Then from the first part of the problem we have

$$(a + (-b))^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k b^{n-k} = 0$$

since $k \geq \alpha$ or $n - k \geq \beta$ (otherwise $n = k + (n - k) < \alpha + \beta$), so that $a^k = 0$ or $b^{n-k} = 0$. Thus N is a subgroup.

To show that N is an ideal, let $r \in R$ and $a \in N$. Suppose that $a^n = 0$. Then $(ra)^n = r^n a^n = r^n \cdot 0 = 0$, so that $ra \in N$. □

2
25 points

3. (25 points) • Let D be an integral domain, and suppose that for every descending chain of ideals in D

$$\cdots \subseteq I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq D$$

there is a positive integer n such that $I_m = I_n$ for all $m \geq n$. *Show that D is a field.*

SOLUTION

Solution. Let $0 \neq x \in D$, and consider the chain of ideals

$$\cdots \subseteq (x^4) \subseteq (x^3) \subseteq (x^2) \subseteq (x)$$

Then there is some positive integer n such that $(x^{n+1}) = (x^n)$. In particular, $x^n \in (x^{n+1})$, so that by definition there exists $y \in D$ such that $x^n = yx^{n+1}$. In other words, $x^n - yx^{n+1} = 0$, or,

$$(1 - yx)x^n = 0.$$

Since we are in an integral domain, and $x \neq 0$, we have that $x^n \neq 0$, and finally that $1 - yx = 0$, so that x is a unit. Since we have shown that every nonzero element of D is a unit, we have that D is a field. $\square$

3
25 points

4. (25 points) • *Show that if F, E, and K are fields with $F \leq E \leq K$, then K is algebraic over F if and only if K is algebraic over E, and E is algebraic over F.* (You must *not* assume the extensions are finite.)

SOLUTION

Solution. This is Fraleigh Exercise 31.31. The solution is available on the course webpage. □

4
25 points