HOMEWORK EXAMPLE

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1. Exercises 9

Exercise 1 (# 9.33). Consider S_n for a fixed $n \geq 2$, and let σ be a fixed odd permutation. The problem asks us to show that every odd permutation in S_n is a product of σ and some permutation in A_n.

Proof. Let σ' be an odd permutation in S_n. We must show that there exists an even permutation $\mu \in A_n$ such that $\sigma' = \sigma\mu$. Indeed, we may take $\mu = \sigma^{-1}\sigma'$, since as the product of two odd permutations, it is an even permutation, and

$$\sigma' = \sigma(\sigma^{-1}\sigma').$$

□

For completeness, let's prove directly that $\sigma^{-1}\sigma'$ is even. From the definition of an odd permutation, there exist a finite number of transpositions $\tau_1, \ldots, \tau_m$ for some odd $m \in \mathbb{N}$ such that

$$\sigma = \tau_1 \ldots \tau_m.$$

Similarly, since σ' is also an odd permutation, there exist a finite number of transpositions $\tau'_1, \ldots, \tau'_\ell$ for some odd $\ell \in \mathbb{N}$ such that $\sigma' = \tau'_1 \ldots \tau'_\ell$. Consider now the permutation

$$\mu = \sigma^{-1}\sigma'.$$

I claim that this lies in A_n. Indeed we have

$$\mu = \sigma^{-1}\sigma' = \underbrace{\tau_m \ldots \tau_1 \tau'_1 \ldots \tau'_\ell}_{m+\ell}.$$

The sum of two odd numbers is even, and so it follows that this is an even permutation.

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