Exercise 9.18

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 9.18 from Fraleigh [Fra03, §9]:

Exercise 9.18. Find the maximum possible order for an element of S_{15}.

Solution. We claim that the maximum possible order for an element of S_{15} is 105.

To see this recall that any element $\sigma \in S_{15}$ can be written as a product of disjoint cycles. If $\sigma_1, \dots, \sigma_r$ are disjoint cycles, then $|\sigma_1 \cdots \sigma_r| = \operatorname{lcm}(|\sigma_1|, \dots, |\sigma_r|)$. In addition, any element $\sigma \in S_{15}$ of maximum possible order can be written as a product of disjoint cycles $\sigma_1 \cdots \sigma_r$ where

$$\sum_{i=1}^{r} |\sigma_i| = 15.$$

In other words, among all partitions $(d_1, \dots, d_r)$ of 15 (i.e., natural numbers $1 \le d_1 \le \cdots \le d_r \le 15$ with $\sum_{i=1}^r d_i = 15$), we want to know what is the maximum of $\operatorname{lcm}(d_1, \dots, d_r)$.

We claim that the maximum is 105, corresponding to the partition $(3, 5, 7)$, which for instance would correspond to the element

$$\sigma = (1,2,3)(4,5,6,7,8)(9,10,11,12,13,14,15) \in S_{15}.$$

We will argue by considering the maximal element of the partition, d_r. For instance, if $d_r = 15$, then the partition is (15), and then the least common multiple is 15. If $d_r = 14$, then the partition is $(1, 14)$ and then the least common multiple is 14. If $d_r = 13$, then the partition is either $(2, 13)$ or $(1, 1, 13)$, and then the maximum of the least common multiples is 26. If $d_r = 12$, then the partition is $(3, 12)$, or $(1, 2, 12)$, or $(1, 1, 1, 12)$, and the maximum of the least common multiples is 12. If $d_r = 11$, then we have $(4, 11)$, or $(1, 3, 11)$, or $(1, 1, 2, 11)$, or $(1, 1, 1, 1, 11)$, in which case the maximum is 44. If $d_r = 10$, then we have $(5, 10)$, or $(1, 4, 10)$, or $(2, 3, 10)$, or $(1, 1, 3, 10)$, or, $(1, 1, 1, 2, 10)$, or $(1, 1, 1, 1, 10)$, in which case the maximum is 30. If $d_r = 9$, then we have

Date: August 8, 2021.

$(6,9)$, or $(1,5,9)$, or $(2,4,9)$, or $(1,1,4,9)$, or $(1,2,3,9)$, or $(1,1,1,3,9)$, or $(2,2,2,9)$, or $(1,1,2,2,9)$, or $(1,1,1,1,2,9)$, or $(1,1,1,1,1,1,9)$, in which case the maximum is 45. Arguing similarly for $d_r = 8,7,6,5,4,3,2,1$, gives the assertion. $\square$

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu