Exercise 6.50

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 6.50 from Fraleigh [Fra03, §6]:

Exercise 6.50. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the *unique* such element. Show that $ax = xa$ for all $x \in G$. [*Hint:* Consider $(xax^{-1})^2$.]

Solution. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the *unique* such element. Let $x \in G$. We will show $xa = ax$.

To do this consider the element $(xax^{-1})^2$. We have

$$\begin{aligned}
(xax^{-1})^2 &= (xax^{-1})(xax^{-1}) \\
&= xa(x^{-1}x)ax^{-1} && \text{Associativity} \\
&= xaax^{-1} \\
&= xx^{-1} && |\langle a \rangle| = 2 \implies a^2 = e \\
&= e
\end{aligned}$$

Therefore, since a is the unique element of G that generates a cyclic subgroup of order 2, we must have

$$xax^{-1} = e \quad \text{or} \quad xax^{-1} = a.$$

In the first case, multiplying by x on the right, we have $xa = x$, then then multiplying by x^{-1} on the left, we have $a = e$, which is not possible since a generates a subgroup of order 2.

In the second case, multiplying by x on the right gives us that

$$xa = ax.$$

□

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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