Exercise 4.31

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 4.31 from Fraleigh [Fra03, §4]:

Exercise 4.31. If $*$ is a binary operation on a set S, an element x of S is an **idempotent for** $*$ if $x * x = x$. Prove that a group has exactly one idempotent element.

Solution. Suppose that $\langle S, * \rangle$ is a group with identity element e. By the definition of the identity element, we have $e * e = e$, so that e is an idempotent element. Now suppose that $x \in S$ is an arbitrary idempotent element; i.e.,

$$x * x = x.$$

We may multiply both sides on the right by x^{-1} to obtain

$$(x * x) * x^{-1} = x * x^{-1}.$$

Using the associative property, we may write this as

$$x * (x * x^{-1}) = x * x^{-1}.$$

From the definition of an inverse element, this gives us

$$x * e = e.$$

Using the definition of the identity, we have

$$x = e.$$

Thus the group has exactly one idempotent element, namely, the identity element. □

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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