**Exercise 3.26**

**Abstract Algebra 1**
**MATH 3140**

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 3.26 from Fraleigh [Fra03, §3]:

**Exercise 3.26.** Recall that if $f : A \to B$ is a one-to-one function mapping $A$ onto $B$, then the element $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, then $\phi^{-1}$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$.

---

For this exercise, we will want to use the following basic fact about inverse functions:

**Lemma 0.1.** *Let $A$ and $B$ be sets, and let $f : A \to B$ be a function. Then $f$ is one-to-one and onto if and only if there exists a function $f^{-1} : B \to A$ such that for all $a \in A$ we have $f^{-1}(f(a)) = a$, and for all $b \in B$ we have $f(f^{-1}(b)) = b$.*

*Proof.* This is a fact you should know from MATH 2001, and it would be a good exercise, to check that you recall the material from that class, to prove this lemma. Recall that $f^{-1}$ is called the inverse function of $f$. □

---

*Solution to Exercise 3.26.* Let $\langle S, * \rangle$ and $\langle S', *' \rangle$ be binary structures, and let $\phi : S \to S'$ be an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$. By definition (see [Fra03, Def. 3.7, p.29]), $\phi : S \to S'$ is a one-to-one function mapping $S$ onto $S'$, such that for all $x, y \in S$:

$$\phi(x * y) = \phi(x) *' \phi(y).$$

As indicated in the statement of the problem (i.e., using Lemma 0.1), since $\phi$ is a one-to-one and onto fuction, $\phi$ has an inverse

$$\phi^{-1} : S' \longrightarrow S.$$

*Date*: August 8, 2021.

Note that in particular, for any $z' \in S'$, we have (see Lemma 0.1)

$$\phi(\phi^{-1}(z')) = z'. \tag{0.1}$$

The problem asks us to show that $\phi^{-1} : S' \to S$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$. In other words, it asks us to show that $\phi^{-1}$ is a one-to-one function mapping $S'$ onto $S$, such that for all $x', y' \in S'$:

$$\phi^{-1}(x' *' y') = \phi^{-1}(x') * \phi^{-1}(y').$$

We already know that $\phi^{-1}$ is a one-to-one function mapping $S'$ onto $S$ (for instance, apply Lemma 0.1 to $\phi^{-1} : S' \to S$, and use $\phi$ to arrive at the implication ( $\Longleftarrow$ ) of the lemma). So all that remains is to check that for all $x', y' \in S'$:

$$\phi^{-1}(x' *' y') = \phi^{-1}(x') * \phi^{-1}(y'). \tag{0.2}$$

So, let $x', y' \in S'$. Since $\phi$ is one-to-one, in order to show that (0.2) holds, it suffices to show that

$$\phi(\phi^{-1}(x' *' y')) = \phi(\phi^{-1}(x') * \phi^{-1}(y')). \tag{0.3}$$

For this, considering first the left hand side, and then the right hand side, we have that

$$\begin{aligned}
\phi(\phi^{-1}(x' *' y')) &= x' *' y' && \text{((0.1), or Lemma 0.1)} \\
\phi(\phi^{-1}(x') * \phi^{-1}(y')) &= \phi(\phi^{-1}(x')) *' \phi(\phi^{-1}(y')) && (\phi \text{ is an isomorphism}) \\
&= x' *' y' && \text{((0.1), or Lemma 0.1)}
\end{aligned}$$

Thus, both sides of (0.3) are equal, and so we have shown that $\phi$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$. □

## References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

*Email address*: casa@math.colorado.edu