Exercise 31.31

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 31.31 from Fraleigh [Fra03, §31]:

Exercise 31.31. Show that if F, E, and K are fields with $F \leq E \leq K$, then K is algebraic over F if and only if E is algebraic over F, and K is algebraic over E. (You must *not* assume the extensions are finite.)

Solution. First assume that K is algebraic over F. We will start by showing that K is algebraic over E. To this end, let $\alpha \in K$. We know that α is algebraic over F, so it satisfies a monic polynomial $f(x) \in F[x]$. But since $F[x] \subseteq E[x]$, we see that α satisfies a monic polynomial over $E[x]$, and so α is algebraic over E. Next let us show that E is algebraic over F. To this end, let $\alpha \in E$. Since $E \subseteq K$, and K is algebraic over F, we see that α satisfies a monic polynomial over F, and we are done.

Next let us assume that K is algebraic over E, and E is algebraic over F, and let us show that K is algebraic over F. So let $\alpha \in K$. We know that K is algebraic over E, so that α satisfies a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in E[x]. \tag{0.1}$$

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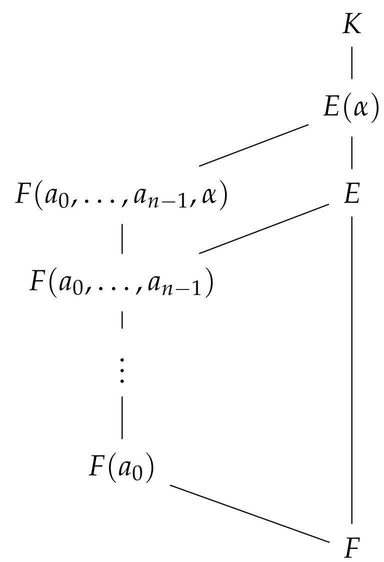
Then we consider the tower of field extensions



Since E is algebraic over F, the extensions $F \leq F(a_0) \leq \cdots \leq F(a_0, \ldots, a_{n-1})$ are algebraic (for instance, this follows from the first paragraph of this solution). By [Fra03, Theorem 30.23], every simple algebraic extension is finite, and by [Fra03, Corollary 31.6], compositions of finite extensions are finite, and so in conclusion, $F(a_0, \ldots, a_{n-1})$ is finite over F (you can also just use [Fra03, Theorem 31.11]).

Finally, the fact that α satisfies the monic polynomial (0.1) implies that α is algebraic over $F(a_1, \ldots, a_{n-1})$. In other words, the extension $F(a_0, \ldots, a_{n-1}) \leq F(a_0, \ldots, a_{n-1}, \alpha)$ is a simple algebraic extension, and is therefore finite. In summary, the extension $F \leq F(a_0, \ldots, a_{n-1}, \alpha)$ is finite [Fra03, Corollary 31.6], and is therefore algebraic by [Fra03, Theorem 31.3]. Thus α is algebraic over F, and we are done. □

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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