

Exercise 27.2

Abstract Algebra 1

MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 27.2 from Fraleigh [Fra03, §27]:

Exercise 27.2. Find all prime ideals and all maximal ideals of \mathbb{Z}_{12} .

Solution. The ideals of \mathbb{Z}_{12} are:

$$\langle 0 \rangle = \{0\}, \langle 1 \rangle = \mathbb{Z}_{12}, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle, \langle 6 \rangle.$$

One can confirm this by seeing that this is a full list of subgroups of \mathbb{Z}_{12} , and then checking that these are again ideals (they are the principal ideals generated by the indicated element; see [Fra03, Definition 27.21]).

Of these ideals, the prime ideals are

$$\langle 2 \rangle, \langle 3 \rangle.$$

Indeed, from [Fra03, Theorem 27.15], we need to check which factor rings are integral domains. We have $\mathbb{Z}_{12}/\langle 0 \rangle = \mathbb{Z}_{12}$ is not an integral domain, we have $\mathbb{Z}_{12}/\langle 1 \rangle = 0$ is not an integral domain, we have $\mathbb{Z}_{12}/\langle 4 \rangle \cong \mathbb{Z}_4$ is not an integral domain, and we have $\mathbb{Z}_{12}/\langle 6 \rangle \cong \mathbb{Z}_6$ is not an integral domain. On the other hand, $\mathbb{Z}_{12}/\langle 2 \rangle \cong \mathbb{Z}_2$ is a field (which is an integral domain), and $\mathbb{Z}_{12}/\langle 3 \rangle \cong \mathbb{Z}_3$ is a field.

Both of the prime ideals

$$\langle 2 \rangle, \langle 3 \rangle$$

are maximal ideals. Indeed, from [Fra03, Theorem 27.9], we need to check which factor rings are fields. But we have already done this in the previous paragraph. \square

Remark 0.1. As an alternative approach, we can start by listing all the ideals in \mathbb{Z}_{12} . These are $\langle 0 \rangle = \{0\}$, $\langle 1 \rangle = \{0, 1, 2, \dots, 12\}$, $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$, $\langle 3 \rangle = \{0, 3, 6, 9\}$, $\langle 4 \rangle = \{0, 4, 8\}$, and

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$\langle 6 \rangle = \{0, 6\}$. From this list we can see that the maximal ideals are $\langle 2 \rangle$ and $\langle 3 \rangle$ (they are the only proper ideals that have no other proper ideals containing them). All of the other proper ideals can be checked to not be prime directly: $3 \cdot 4 = 0 \in \langle 0 \rangle$, but $3, 4 \notin \langle 0 \rangle$, and similarly, $2 \cdot 2 = 4 \in \langle 4 \rangle$, $2 \cdot 3 = 6 \in \langle 6 \rangle$.

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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