Exercise 26.21

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 26.21 from Fraleigh [Fra03, §26]:

Exercise 26.21. Let R and R' be rings and let $\phi : R \to R'$ be a ring homomorphism such that $\phi[R] \neq \{0'\}$. Show that if R has unity 1 and R' has no 0 divisors, then $\phi(1)$ is unity for R'.

Solution. Let R and R' be rings and let $\phi : R \to R'$ be a ring homomorphism such that $\phi[R] \neq \{0'\}$. Suppose further that R has unity 1, and *assume only the weaker hypothesis* that $\phi(1)$ is not a zero divisor in R' (which of course is true if R' has no 0 divisors). We will show that $\phi(1)$ is unity for R'.

The first observation is that

$$\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1), \tag{0.1}$$

since ϕ is a homomorphism of rings. The second observation is that $\phi(1) \neq 0'$, since otherwise, for any $r \in R$ we would have $\phi(r) = \phi(1 \cdot r) = \phi(1)\phi(r) = 0' \cdot r = 0'$, contradicting the assumption that $\phi[R] \neq \{0'\}$.

Now, with these observations, to show that $\phi(1)$ is a left multiplicative identity, we want to show that for all $r' \in R'$, we have

$$\phi(1)r' = r'. \tag{0.2}$$

This is equivalent to showing

$$\phi(1)r' - r' = 0, \tag{0.3}$$

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which, since $\phi(1) \neq 0'$ and is not a zero divisor, is equivalent to showing

$$\phi(1)(\phi(1)r' - r') = 0. \tag{0.4}$$

In other words, to show that $\phi(1)$ is a left multiplicative identity for R', it suffices to show that (0.4) holds for all $r' \in R$. We now prove this:

$$\begin{aligned} \phi(1)(\phi(1)r' - r') &= \phi(1)\phi(1)r' - \phi(1)r' \\ &= \phi(1)r' - \phi(1)r' && \text{(using (0.1))} \\ &= 0. \end{aligned}$$

This completes the proof that $\phi(1)$ is a left multiplicative identity. A similar argument shows that it is also a right multiplicative identity, and therefore unity in R'. □

Remark 0.1. We also have the related result: if R is a ring with unity 1 and $\phi : R \to R'$ is a surjective ring homomorphism, then $\phi(1)$ is unity for R'. Indeed, for any $r' \in R'$, there exists $r \in R$ such that $\phi(r) = r'$, and then we have $\phi(1)r' = \phi(1)\phi(r) = \phi(1 \cdot r) = \phi(r) = r'$, and similarly, one can show $r'\phi(1) = r'$.

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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