Exercise 23.34

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 23.34 from Fraleigh [Fra03, §23]:

Exercise 23.34. Show that for p a prime, the polynomial $x^p + a$ in $\mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$.

Solution. From the Factor Theorem [Fra03, Corollary 23.3], it suffices to show that for any $a \in \mathbb{Z}_p$, the polynomial $x^p + a$ has a root (or "zero") α in $\mathbb{Z}_p$ (since then $x - \alpha$ will be a factor of $x^p + a$, and as $p \geq 2$, this implies that $x^p + a$ is reducible). In fact $x = -a$ (i.e., $x = p - a$) is a root of the polynomial $x^p + a$ for any $a \in \mathbb{Z}_p$. This follows from Fermat's Little Theorem [Fra03, Corollary 20.2], which states that $x^p = x$ in $\mathbb{Z}_p$, so that $(-a)^p + a = -a + a = 0 \in \mathbb{Z}_p$. $\square$

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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