Exercise 1.17

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 1.17 from Fraleigh [Fra03, §1]:

Exercise 1.17. Find all solutions in $\mathbb{C}$ of the equation $z^4 = -1$.

Solution. First observe that $\zeta = e^{i\pi/4}$ is a solution to the equation $z^4 = -1$, since $\zeta^4 = (e^{i\pi/4})^4 = e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$. One can check that $\zeta, i\zeta, -\zeta, -i\zeta$ are also solutions, and are not equal to one another (e.g., $(-i\zeta)^4 = (-i)^4\zeta^4 = 1\cdot(-1) = -1$). By the Fundamental Theorem of Algebra, the equation $z^4 = -1$ has at most four solutions, and so we are done. $\square$

Remark 0.1. One can also solve the problem as follows. Since $1 = |-1| = |z^4| = |z|^4$, we know that $|z| = 1$, so that any solution of the equation $z^4 = -1$ will be of the form $z = e^{i\theta}$. Then we have $z^4 = (e^{i\theta})^4 = e^{i4\theta}$, and this must equal $-1 = e^{i\pi}$, so that putting this together we have $4\theta = \pi + 2\pi n$ ($n \in \mathbb{Z}$). Thus $\theta = \frac{\pi}{4} + \frac{\pi}{2}n$ ($n \in \mathbb{Z}$). In other words

$$\theta \in \{\dots, \frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + \frac{3\pi}{2}, \frac{\pi}{4} + 2\pi, \cdots\}$$

Since $e^{i\theta} = e^{i(\theta+2\pi)}$, and $e^{i\pi/2} = i$, we see the solutions are $z = e^{i\pi/4}, ie^{i\pi/4}, -e^{i\pi/4}, -ie^{i\pi/4}$.

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REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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