

Exercise 0.18

Abstract Algebra 1

MATH 3140

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ABSTRACT. This is Exercise 0.18 from Fraleigh [Fra03, §0]:

Exercise 0.18. For any set A , finite or infinite, let B^A be the set of all functions mapping A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set $\mathcal{P}(A)$. [Hint: Each element of B^A determines a subset of A in a natural way.]

Solution. To show that the cardinality of B^A is the same as the cardinality of the set $\mathcal{P}(A)$, we need to construct a one-to-one and onto function

$$\phi : B^A \longrightarrow \mathcal{P}(A).$$

We define ϕ as follows. Given a function $f : A \rightarrow \{0, 1\}$, we define

$$\phi(f) := \{a \in A : f(a) = 1\} \subseteq A.$$

Now we must show it is one-to-one and onto. First let us show it is one-to-one. So suppose that $f, g \in B^A$, and $\phi(f) = \phi(g)$. In other words,

$$\phi(f) = \{a \in A : f(a) = 1\} = \phi(g) = \{a \in A : g(a) = 1\}.$$

Since any function $A \rightarrow \{0, 1\}$ is determined by the elements of A that it sends to 1 (it must send the remaining elements to 0), we see that $f = g$. Thus ϕ is one-to-one.

Now let us show that ϕ is onto. Let $S \subseteq A$. Then let $1_S : A \rightarrow \{0, 1\}$ be the function defined by

$$1_S(a) = \begin{cases} 1, & a \in S \\ 0, & a \notin S. \end{cases}$$

Then $\phi(1_S) = S$, and therefore ϕ is onto. □

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Remark 0.1. As an alternate approach, to show that ϕ is one-to-one and onto, it suffices to construct an inverse function

$$\psi : \mathcal{P}(A) \longrightarrow B^A;$$

i.e., a function ψ as above such that for all $f \in B^A$ we have $\psi(\phi(f)) = f$, and for all $S \subseteq A$ we have $\phi(\psi(S)) = S$. We construct ψ as follows. Given a subset $S \subseteq A$, we define $\psi(S) : A \rightarrow \{0,1\}$ by the rule

$$\psi(S)(a) = \begin{cases} 1, & a \in S \\ 0, & a \notin S. \end{cases}$$

You can check that ψ is an inverse function for ϕ . (You can also see that the function 1_S in the solution above is equal to the function $\psi(S)$.)

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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