Midterm 2

Linear Algebra

MATH 2130

Spring 2021

Friday March 19, 2021

NAME: **Enter your name here** ____________________

PRACTICE EXAM

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

- This exam is closed book.
- You may use only paper and pencil.
- You may not use any other resources whatsoever.
- You will be graded on the clarity of your exposition.

1. (10 points) • Let $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, and suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces. Recall that there is set $V_1 \times V_2$, called the product of V_1 and V_2, whose elements consist of the ordered pairs (v_1, v_2) such that $v_1 \in V_1$ and $v_2 \in V_2$.

Define a map of sets

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$

$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and a map of sets

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$

$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Show that the triple $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$ *is a K-vector space*. We call this the product of the vector spaces $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$.

2. (10 points) • Find the determinant of each of the following matrices.

(a) $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

3. (10 points) • Let $V = \mathbb{R}[x]$ be the real vector space of real polynomial functions. Let

$$D : V \to V$$

$$p(x) \mapsto p'(x)$$

be the derivative map; i.e., $D(p(x)) = p'(x)$ for all polynomials $p(x) \in V$. Let

$$E : V \to V$$

$$p(x) \mapsto \int_0^x p(t)\, dt$$

be the integration map that sends a polynomial $p(x) \in V$ to the polynomial $q(x) \in V$ given by the rule $q(x) = \int_0^x p(t)\, dt$. It is a fact (which you can use without proof) that D and E are linear maps.

(a) *Show that D is surjective, but not injective.*

(b) *Show that E is injective, but not surjective.*

4. (10 points) • Suppose we have a two state Markov chain with stochastic matrix

$$P = \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix}$$

Given the probability vector $v = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$*, find* $\lim_{n\to\infty} P^n v$.

5. (10 points) • Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

(a) *Find the characteristic polynomial $p_A(t)$ of A.*

(b) *Find the eigenvalues of A.*

(c) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

(d) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

6. (10 points) • Consider the following matrix:

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

(a) *What is the sum of the roots of the characteristic polynomial of B?*

(b) *What is the product of the roots of the characteristic polynomial of B?*

(c) *Are all of the roots of the characteristic polynomial of B real?*

7. (10 points) • Consider the two dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

(a) *Is the origin an attractor, repeller, or saddle point?*

(b) *Find the directions of greatest attraction or repulsion.*

8. (10 points) • **TRUE** or **FALSE**:

(a) Suppose A and B are invertible $n \times n$ matrices, and that $AB = BA$. Then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

(b) Let $f : V \to V$ be a linear map of a vector space to itself. If f is surjective, then f is an isomorphism.

(c) Suppose that P is an $n \times n$ matrix with positive entries, such that the column sums are equal to 1. Then $\lim_{n\to\infty} P^n$ exists.

(d) Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \operatorname{Im}(T)$.

(e) If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors.

(f) If v is an eigenvector for an $n \times n$ matrix A with eigenvalue λ, and $r \neq 0$ is a real number, then rv is an eigenvector for A with eigenvalue λ.

(g) Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

(h) For an $n \times n$ matrix A, if $\det(\operatorname{cof} A) = 0$, then $\det A = 0$.

(i) If V is a real vector space, and $W, W' \subseteq V$ are real vector subspaces of V, then $W \cap W'$ is a real vector subspace of V.

(j) The row space of a matrix is the same as the row space of the reduced row echelon form of the matrix.