Final Exam

Linear Algebra

MATH 2130

Spring 2021

Saturday May 1, 2021

NAME: ____

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points:	20	20	20	20	20	20	120
Score:							

- This exam is closed book.
- You may use only paper and pencil.
- You may not use any other resources whatsoever.
- You will be graded on the clarity of your exposition.

1. (20 points) • Let $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, let V_1, V_2, V_1', and V_2' be K-vector spaces, and suppose that $L_1 : V_1 \to V_1'$ and $L_2 : V_2 \to V_2'$ are linear maps of K-vector spaces.

Recall that there is a so-called product linear map of K-vector spaces defined as follows on the products of the K-vector spaces:

$$L = L_1 \times L_2 : V_1 \times V_2 \longrightarrow V_1' \times V_2'$$

$$L(v_1, v_2) = (L_1(v_1), L_2(v_2)).$$

If L_1 and L_2 are isomorphisms, show that L is also an isomorphism.

SOLUTION

Solution. We can show that the linear map L is an isomorphism by showing that it is bijective (injective and surjective).

To show L is injective, it is equivalent to show that $\ker L$ is trivial; i.e., $\ker L = \{(\mathscr{O}_{V_1}, \mathscr{O}_{V_2})\}$, where $\mathscr{O}_{V_1}$ is the identity element for V_1, and similarly for $\mathscr{O}_{V_2}$ — here we are using that the identity element for $V_1 \times V_2$ is $(\mathscr{O}_{V_1}, \mathscr{O}_{V_2})$, a fact we have shown before.

To this end, let $\mathscr{O}_{V_1'}$ be the identity element for V_1', and similarly for $\mathscr{O}_{V_2'}$, so that $(\mathscr{O}_{V_1'}, \mathscr{O}_{V_2'})$ is the identity element for $V_1' \times V_2'$. Then $L(v_1, v_2) = (L_1(v_1), L_2(v_2)) = (\mathscr{O}_{V_1'}, \mathscr{O}_{V_2'})$ if and only if $v_1 = \mathscr{O}_{V_1}$, and $v_2 = \mathscr{O}_{V_2}$, since L_1 and L_2 are injective. Thus $\ker L$ is trivial, and so L is injective.

To show L is surjective, we argue as follows. Suppose that $(v_1', v_2') \in V_1' \times V_2'$. Then, since L_1 and L_2 are surjective, there exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $L_1(v_1) = v_1'$ and $L_2(v_2) = v_2'$. Then $L(v_1, v_2) = (L_1(v_1), L_2(v_2)) = (v_1', v_2')$, so that L is surjective.

In summary, having established that L is a bijective linear map, we can conclude L is an isomorphism.

□

ANOTHER SOLUTION

Another solution. In this solution, we will show that L is an isomorphism by constructing an inverse linear map

$$L^{-1} : V_1' \times V_2' \to V_1 \times V_2.$$

In other words, we will construct a linear map $L^{-1} : V_1' \times V_2' \to V_1 \times V_2$, and then show that $L^{-1}L = \mathrm{Id}_{V_1 \times V_2}$ and $LL^{-1} = \mathrm{Id}_{V_1' \times V_2'}$. Recall that $\mathrm{Id}_{V_1 \times V_2}$ is the identity map on $V_1 \times V_2$, and is defined by the rule that for all $(v_1, v_2) \in V_1 \times V_2$, we have $\mathrm{Id}_{V_1 \times V_2}(v_1, v_2) = (v_1, v_2)$, and similarly for $\mathrm{Id}_{V_1' \times V_2'}$.

To construct L^{-1}, and establish the properties above, we start with the observation that since L_1 and L_2 are isomorphisms, they admit inverse linear maps $L_1^{-1} : V_1' \to V_1$ and $L_2^{-1} : V_2' \to V_2$, respectively. By definition, these satisfy the conditions: $L_1^{-1}L_1 = \mathrm{Id}_{V_1}$, $L_1L_1^{-1} = \mathrm{Id}_{V_1'}$, $L_2^{-1}L_2 = \mathrm{Id}_{V_2}$, and $L_2L_2^{-1} = \mathrm{Id}_{V_2'}$.

We define L^{-1} to be the product of the maps L_1^{-1} and L_2^{-1}:

$$L^{-1} = L_1^{-1} \times L_2^{-1} : V_1' \times V_2' \longrightarrow V_1 \times V_2$$
$$L^{-1}(v_1', v_2') = (L_1^{-1}(v_1'), L_2^{-1}(v_2')).$$

Then for any $(v_1, v_2) \in V_1 \times V_2$ we have

$$\begin{aligned} L^{-1}L(v_1, v_2) &= L^{-1}(L_1(v_1), L_2(v_2)) && \text{Def. of } L \\ &= (L_1^{-1}L_1(v_1), L_2^{-1}L_2(v_2)) && \text{Def. of } L^{-1} \\ &= (\mathrm{Id}_{V_1}(v_1), \mathrm{Id}_{V_2}(v_2)) && \text{See above} \\ &= (v_1, v_2) && \text{Def. of } \mathrm{Id}_{V_1}, \mathrm{Id}_{V_2} \end{aligned}$$

Therefore, $L^{-1}L = \mathrm{Id}_{V_1 \times V_2}$.

Similarly, for any $(v_1', v_2') \in V_1' \times V_2'$ we have

$$\begin{aligned} LL^{-1}(v_1', v_2') &= L(L_1^{-1}(v_1'), L_2^{-1}(v_2')) && \text{Def. of } L^{-1} \\ &= (L_1L_1^{-1}(v_1'), L_2L_2^{-1}(v_2')) && \text{Def. of } L \\ &= (\mathrm{Id}_{V_1'}(v_1'), \mathrm{Id}_{V_2'}(v_2')) && \text{See above} \\ &= (v_1', v_2') && \text{Def. of } \mathrm{Id}_{V_1'}, \mathrm{Id}_{V_2'} \end{aligned}$$

Therefore, $LL^{-1} = \mathrm{Id}_{V_1' \times V_2'}$.

This completes the proof that L^{-1} is an inverse to L, and consequently, that L is an isomorphism. $\square$

1
20 points

2. (20 points) $\bullet$ Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ *spanned by* $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, *and* $\mathbf{x}_4$.

SOLUTION

Solution. An orthonormal basis is given by

$$\boxed{\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}} \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}}$$

We start by finding an orthogonal basis. We have

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 . \mathbf{y}_1}{\mathbf{y}_1 . \mathbf{y}_1} \mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ -1/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

For simplicity, we will take

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

We have

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 . \mathbf{y}_1}{\mathbf{y}_1 . \mathbf{y}_1}\mathbf{y}_1 - \frac{\mathbf{x}_3 . \mathbf{y}_2}{\mathbf{y}_2 . \mathbf{y}_2}\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{15}\begin{bmatrix} 0 \\ 0 \\ 15 \\ 15 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} 2 \\ -4 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 \\ -9 \\ 9 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Again for simplicity we take

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Note that since $\mathbf{x}_4 = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3$, we see that $\mathbf{x}_4$ is in the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, so if we perform Gram–Schmidt to $\mathbf{x}_4$, we will get $\mathbf{y}_4 = 0$. I omit the computation here for brevity.

Therefore, an orthogonal basis for the span of $\mathbf{x}_1, \dots, \mathbf{x}_4$ is given by

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Consequently, an orthonormal basis is given by

$$\mathbf{u}_1 = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

□

2
20 points

3. • Consider the following real matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) (4 points) *Find the characteristic polynomial $p_A(t)$ of A.*

(b) (4 points) *Find the eigenvalues of A.*

(c) (4 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

(d) (4 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

(e) (4 points) *Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix $U \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $U^T AU$ is diagonal. If not, explain.*

SOLUTION

Solution to (a). The characteristic polynomial of A is:

$$\boxed{p_A(t) = \det(tI - A) = t^3 - 11t^2 + 36t - 36}$$

If you used the textbook's convention, you will get $p_A(t) = \det(A - tI) = 36 - 36t + 11t^2 - t^3$; that is also fine.

Here is the computation.

$$\det(tI - A) = \begin{vmatrix} t-3 & +1 & -1 \\ +1 & t-5 & +1 \\ -1 & +1 & t-3 \end{vmatrix}$$

$$\begin{aligned}
&= (t-3)[(t-5)(t-3) - (1)(1)] - (1)[(t-3) - (1)(-1)] + (-1)[(1)(1) - (t-5)(-1)] \\
&= (t-3)[t^2 - 8t + 15 - 1] - [t - 3 + 1] - [1 + t - 5] \\
&= (t-3)[t^2 - 8t + 14] - [t-2] - [t-4] \\
&= [t^3 - 8t^2 + 14t - 3t^2 + 24t - 42] - 2t + 6 \\
&= t^3 - 11t^2 + 36t - 36
\end{aligned}$$

$\square$

Solution to (b). The eigenvalues of A are

$$\boxed{\lambda = 6, 3, 2}$$

The computation is as follows. By trying, $0, \pm 1, \pm 2$, we see that

$$p_A(2) = 0.$$

Thus we have

$$\begin{aligned} p_A(t) &= t^3 - 11t^2 + 36t - 36 \\ &= (t-2)(t^2 - 9t + 18) \\ &= (t-2)(t-3)(t-6) \end{aligned}$$

Therefore, the real roots of $p_A(t)$ are $\lambda = 6, 3, 2$. $\square$

Solution to (c). A basis for each eigenspace is:

$$\boxed{E_6 \leftrightarrow \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad E_3 \leftrightarrow \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad E_2 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.}$$

The computation is as follows. We start with E_6. We want to find a basis for the kernel of

$$6I - A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ 3 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We then adjust the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ {\color{blue}0} & {\color{blue}0} & {\color{red}-1} \end{bmatrix}$$

The last column, with the new red ${\color{red}-1}$, gives the basis element we want.

Next we consider E_3. We want to find a basis for the kernel of

$$3I - A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We then adjust the matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ {\color{blue}0} & {\color{blue}0} & {\color{red}-1} \end{bmatrix}$$

The last column, with the new red ${\color{red}-1}$, gives the basis element we want.

Finally we consider E_2. We want to find a basis for the kernel of

$$2I - A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We then adjust the matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} \end{bmatrix}$$

The last column, with the new red $\color{red}{-1}$, gives the basis element we want. □

Solution to (d). $\boxed{\text{Yes, } A \text{ is diagonalizable,}}$ since every symmetric matrix is diagonalizable.

We can use the matrix with columns given by the basis elements for the eigenspaces that we just computed. In other words, we may take

$$\boxed{S = \begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}.}$$

□

Solution to (e). $\boxed{\text{Yes, } A \text{ is diagonalizable with orthogonal matrices,}}$ since every symmetric matrix is diagonalizable with orthogonal matrices.

We can use the matrix with columns given by the orthonormalized basis elements for the eigenspaces that we just computed (i.e., obtained by applying Gram–Schmidt to each basis for each eigenspace). Since we only have one dimensional eigenspaces, the Gram–Schmidt process simply divides each basis vector by its length, and so we may take

$$\boxed{U = \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}.}$$

□

3
20 points

4. (20 points) • Let $\mathbb{P}_3$ be the real vector space of polynomials of degree at most 3 (my notation for this vector space has been $\mathbb{R}[t]_3$, but I am using the textbook's notation here). A basis of $\mathbb{P}_3$ is given by the polynomials $1, t, t^2, t^3$.

We have seen that there is an inner product on $\mathbb{P}_3$ given by evaluation at -2, -1, 1, and 2. In other words, given polynomials $p(t), q(t) \in \mathbb{P}_3$, we define the inner product by the rule

$$\begin{aligned}(p(t), q(t)) &:= (p(-2), p(-1), p(1), p(2)).(q(-2), q(-1), q(1), q(2)) \\ &= p(-2)q(-2) + p(-1)q(-1) + p(1)q(1) + p(2)q(2).\end{aligned}$$

Let $p_1(t) = t$, and $p_2(t) = t^2$.

Find the best approximation to $p(t) = t^3$ by the polynomials in $\operatorname{Span}\{p_1(t), p_2(t)\}$. In other words, find the polynomial $q(t)$ in the span of $p_1(t)$ and $p_2(t)$, that is closest to the polynomial $p(t)$ with respect to the given inner product on $\mathbb{P}_3$.

SOLUTION

Solution. The best approximation to $p(t) = t^3$ by the polynomials in $\operatorname{Span}\{p_1(t), p_2(t)\}$ is

$$\boxed{q(t) = \frac{17}{5}t.}$$

To show this, we need to compute the orthogonal projection of $p(t) = t^3$ onto $\operatorname{Span}\{p_1(t), p_2(t)\} = \operatorname{Span}\{t, t^2\}$.

First we want to compute an orthonormal basis for $\operatorname{Span}\{t, t^2\}$, which we obtain by performing Gram–Schmidt on the given basis $p_1(t) = t$, $p_2(t) = t^2$.

For this problem, it is convenient to make the following table

	-2	-1	1	2
$p_1(t) = t$	$(-2,$	$-1,$	$1,$	$2)$
$p_2(t) = t^2$	$(4,$	$1,$	$1,$	$4)$
$p(t) = t^3$	$(-8,$	$-1,$	$1,$	$8)$

Then the inner product in $\mathbb{P}_3$ is given by dotting the corresponding vectors above. In other words, we have

$$\begin{aligned}
(p_1(t), p_1(t)) &= (-2, -1, 1, 2).(-2, -1, 1, 2) = 10 \\
(p_1(t), p_2(t)) &= (-2, -1, 1, 2).(4, 1, 1, 4) = 0 \\
(p_1(t), p(t)) &= 34 \\
(p_2(t), p_2(t)) &= 34 \\
(p_2(t), p(t)) &= 0
\end{aligned}$$

Fortunately, we see that the basis $p_1(t)$, $p_2(t)$, is already orthogonal. Thus we can compute the projection $q(t)$ of $p(t)$ onto $\text{Span}\{p_1(t), p_2(t)\}$ as

$$\begin{aligned}
q(t) &= \frac{(p(t), p_1(t))}{(p_1(t), p_1(t))} p_1(t) + \frac{(p(t), p_2(t))}{(p_2(t), p_2(t))} p_2(t) \\
&= \frac{34}{10} p_1(t) + 0 p_2(t) \\
&= \frac{17}{5} t
\end{aligned}$$

□

4
20 points

5. (20 points) • *Maximize the quadratic form*

$$Q(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 2x_1x_3 + 5x_2^2 - 2x_2x_3 + 3x_3^2$$

subject to the constraint that $x_1^2 + x_2^2 + x_3^2 = 1$. In other words, find the maximum of the given quadratic form restricted to the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

[Hint: Compare to the matrix in Problem 3.]

SOLUTION

Solution. The maximum of $Q(x_1, x_2, x_3)$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$ is $\boxed{6}$. Although it is not asked for in this problem, let us note here that this maximum is achieved at $\pm\frac{1}{\sqrt{6}}(-1, 2, -1)$.

To show this, one can see that for $\mathbf{x} = (x_1, x_2, x_3)$, we have

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \qquad \text{where } A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

We already worked out in Problem 3 that the eigenvalues of A are $\lambda = 6, 3, 2$. Therefore the maximum of $Q(x_1, x_2, x_3)$ subject to the condition $x_1^2 + x_2^2 + x_3^2 = 1$ is 6, i.e., the largest eigenvalue of A.

Although it is not asked for in for this problem, we can easily find the points where we achieve this maximum. Indeed, we already worked out in Problem 3 that an orthonormal basis for each eigenspace of A is given by:

$$E_6 \leftrightarrow \mathbf{u}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad E_3 \leftrightarrow \mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad E_2 \leftrightarrow \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Since the maximum of $Q(x_1, x_2, x_3)$ subject to the condition $x_1^2 + x_2^2 + x_3^2 = 1$ is achieved at any unit vector in the eigenspace for A with the largest eigenvalue (i.e., $\lambda = 6$), the maximum occurs at plus and minus the given orthonormal basis vector $\mathbf{u}_1$ for E_6 (since $\dim E_6 = 1$, the vectors $\pm\mathbf{u}_1$ are the only unit vectors in E_6). □

5
20 points

6. • **TRUE** or **FALSE**. You do **not** need to justify your answer.

(a) (2 points) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

TRUE: This is Cauchy–Schwarz.

(b) (2 points) Two vectors in $\mathbb{R}^n$ are orthogonal if their dot product is zero.

TRUE: This was our definition of orthogonal.

(c) (2 points) If $W \subseteq \mathbb{R}^n$ is a vector subspace and $W^\perp$ is the orthogonal complement, then $W \subseteq W^\perp$.

FALSE: For instance, take $n > 0$ and take $W = \mathbb{R}^n$, so that $W^\perp = 0$.

(d) (2 points) If $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, then a least squares solution to the equation $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

TRUE: We showed this in class – this is Theorem 13, p.363 of Lay.

(e) (2 points) For the real vector space $C^0([0,1])$ consisting of continuous functions $f : [0,1] \to \mathbb{R}$ on the closed interval $[0,1]$, the rule

$$(f(t), g(t)) = \int_0^1 f(t)g(t)\, dt$$

defines an inner product on $C^0([0,1])$.

TRUE: We showed this in class, and this is also discussed in §6.7 of Lay.

(f) (2 points) If A is any real matrix, then the matrix A^TA has non-negative eigenvalues.

TRUE: We showed this in class. As a reminder, here is the sketch of the proof. Considering an eigenvector $\mathbf{x}$ for A^TA with eigenvalue λ, one has $0 \leq (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda\|\mathbf{x}\|^2$. Dividing by $\|\mathbf{x}\|^2 > 0$ gives the assertion.

(g) (2 points) Every real square matrix is diagonalizable with orthogonal matrices.

FALSE: There are some matrices that are not diagonalizable at all; e.g., $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(h) (2 points) Given symmetric matrices A and B of the same size, then AB is a symmetric matrix.

FALSE: For instance, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

(i) (2 points) Every quadratic form has a maximum value.

FALSE: Take $Q(x) = x^2$. This quadratic form has no maximum value.

(j) (2 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The angle θ between $\mathbf{x}$ and $\mathbf{y}$ satisfies $\cos\theta = \dfrac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$.

TRUE: This was our definition of the angle between vectors in $\mathbb{R}^n$.

6
20 points