Final Exam

Linear Algebra

MATH 2130

Spring 2021

Saturday May 1, 2021

NAME: ______________________________

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points:	20	20	20	20	20	20	120
Score:							

- This exam is closed book.
- You may use only paper and pencil.
- You may not use any other resources whatsoever.
- You will be graded on the clarity of your exposition.

1. (20 points) • Let $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, let V_1, V_2, V_1', and V_2' be K-vector spaces, and suppose that $L_1 : V_1 \to V_1'$ and $L_2 : V_2 \to V_2'$ are linear maps of K-vector spaces.

Recall that there is a so-called product linear map of K-vector spaces defined as follows on the products of the K-vector spaces:

$$L = L_1 \times L_2 : V_1 \times V_2 \longrightarrow V_1' \times V_2'$$

$$L(v_1, v_2) = (L_1(v_1), L_2(v_2)).$$

If L_1 and L_2 are isomorphisms, show that L is also an isomorphism.

1
20 points

2. (20 points) • Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ *spanned by* $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, *and* $\mathbf{x}_4$.

2
20 points

3. • Consider the following real matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(a) (4 points) *Find the characteristic polynomial $p_A(t)$ of A.*

(b) (4 points) *Find the eigenvalues of A.*

(c) (4 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

(d) (4 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

(e) (4 points) *Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix $U \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $U^T AU$ is diagonal. If not, explain.*

3
20 points

4. (20 points) • Let $\mathbb{P}_3$ be the real vector space of polynomials of degree at most 3 (my notation for this vector space has been $\mathbb{R}[t]_3$, but I am using the textbook's notation here). A basis of $\mathbb{P}_3$ is given by the polynomials $1, t, t^2, t^3$.

We have seen that there is an inner product on $\mathbb{P}_3$ given by evaluation at -2, -1, 1, and 2. In other words, given polynomials $p(t), q(t) \in \mathbb{P}_3$, we define the inner product by the rule

$$\begin{aligned}(p(t), q(t)) &:= (p(-2), p(-1), p(1), p(2)).(q(-2), q(-1), q(1), q(2)) \\ &= p(-2)q(-2) + p(-1)q(-1) + p(1)q(1) + p(2)q(2).\end{aligned}$$

Let $p_1(t) = t$, and $p_2(t) = t^2$.

Find the best approximation to $p(t) = t^3$ *by the polynomials in* $\mathrm{Span}\{p_1(t), p_2(t)\}$. In other words, find the polynomial $q(t)$ in the span of $p_1(t)$ and $p_2(t)$, that is closest to the polynomial $p(t)$ with respect to the given inner product on $\mathbb{P}_3$.

4
20 points

5. (20 points) • *Maximize the quadratic form*

$$Q(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 2x_1x_3 + 5x_2^2 - 2x_2x_3 + 3x_3^2$$

subject to the constraint that $x_1^2 + x_2^2 + x_3^2 = 1$. In other words, find the maximum of the given quadratic form restricted to the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

[Hint: Compare to the matrix in Problem 3.]

5
20 points

6. • **TRUE** or **FALSE**. You do **not** need to justify your answer.

(a) (2 points) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

(b) (2 points) Two vectors in $\mathbb{R}^n$ are orthogonal if their dot product is zero.

(c) (2 points) If $W \subseteq \mathbb{R}^n$ is a vector subspace and $W^{\perp}$ is the orthogonal complement, then $W \subseteq W^{\perp}$.

(d) (2 points) If $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, then a least squares solution to the equation $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

(e) (2 points) For the real vector space $C^0([0,1])$ consisting of continuous functions $f : [0,1] \to \mathbb{R}$ on the closed interval $[0,1]$, the rule

$$(f(t), g(t)) = \int_0^1 f(t)g(t)\, dt$$

defines an inner product on $C^0([0,1])$.

(f) (2 points) If A is any real matrix, then the matrix $A^T A$ has non-negative eigenvalues.

(g) (2 points) Every real square matrix is diagonalizable with orthogonal matrices.

(h) (2 points) Given symmetric matrices A and B of the same size, then AB is a symmetric matrix.

(i) (2 points) Every quadratic form has a maximum value.

(j) (2 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The angle θ between $\mathbf{x}$ and $\mathbf{y}$ satisfies $\cos\theta = \dfrac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$.

6
20 points