Exercise 15.39

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 15.39 from Fraleigh [Fra03, §15]:

Exercise 15.39. Prove that A_n is simple for $n \geq 5$, following the steps and hints given.

a. Show that A_n contains every 3-cycle if $n \geq 3$.

b. Show that A_n is generated by the 3-cycles for $n \geq 3$. [*Hint:* Note that $(a,b)(c,d) = (a,c,b)(a,c,d)$ and $(a,c)(a,b) = (a,b,c)$.]

c. Let r and s be fixed elements of $\{1,2,\ldots,n\}$ for $n \geq 3$. Show that A_n is generated by the n "special" 3-cycles of the form (r,s,i) for $1 \leq i \leq n$ [*Hint:* Show every 3-cycle is the product of "special" 3-cycles by computing

$$(r,s,i)^2, \ \ (r,s,j)(r,s,i)^2, \ \ (r,s,j)^2(r,s,i),$$

and

$$(r,s,i)^2(r,s,k)(r,s,j)^2(r,s,i).$$

Observe that these products give all possibly types of 3-cycles.]

d. Let N be a normal subgroup of A_n for $n \geq 3$. Show that if N contains a 3-cycle, then $N = A_n$. [*Hint:* Show that $(r,s,i) \in N$ implies that $(r,s,j) \in N$ for $j = 1,2,\ldots,n$ by computing

$$((r,s)(i,j))(r,s,i)^2((r,s)(i,j))^{-1}.]$$

e. Let N be a nontrivial normal subgroup of A_n for $n \geq 5$. Show that one of the following cases must hold, and conclude in each case that $N = A_n$.

- **Case I:** N contains a 3-cycle.

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- **Case II:** N contains a product of disjoint cycles, at least one of which has length greater than 3. [*Hint:* Suppose N contains the disjoint product $\sigma = \mu(a_1, a_2, \ldots, a_r)$. Show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it.]
- **Case III:** N contains a disjoint product of the form $\sigma = \mu(a_4, a_5, a_6)(a_1, a_2, a_3)$. [*Hint*: Show $\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1}$ is in N, and compute it.]
- **Case IV:** N contains a disjoint product of the form $\sigma = \mu(a_1, a_2, a_3)$ where μ is a product of disjoint 2-cycles. [*Hint:* Show $\sigma^2 \in N$ and compute it.]
- **Case V:** N contains a product σ of the form $\sigma = \mu(a_3, a_4)(a_1, a_2)$ where μ is a product of an even number of disjoint 2-cycles. [*Hint*: Show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it to deduce that $\alpha = (a_2, a_4)(a_1, a_4)$ is in N. Using $n \geq 5$ for the first time, find $i \neq a_1, a_2, a_3, a_4$ in $\{1, 2, \ldots, n\}$. Let $\beta = (a_1, a_3, i)$. Show that $\beta^{-1}\alpha\beta\alpha \in N$, and compute it.]

Solution. **a.** A_n contains every 3-cycle if $n \geq 3$.

Proof. Let $(a_1, a_2, a_3) \in S_n$ be a 3-cycle. Since $(a_1, a_2, a_3) = (a_1, a_2)(a_3, a_2)$ it follows from the definition that $(a_1, a_2, a_3) \in A_n$. □

b. A_n is generated by the 3-cycles for $n \geq 3$.

Proof. Let $\sigma \in A_n$ be a nontrivial element. By definition there is an expression of σ

$$\sigma = \tau_1\tau_2 \cdots \tau_{2n-1}\tau_{2n}$$

as a composition of transpositions $\tau_1, \ldots, \tau_{2n}$ for some $n \in \mathbb{N}$. Since there are n-pairs of transpositions in the expression, the claim will follow if we can show that for any transpositions $\tau, \hat{\tau} \in S_n$ with $\tau \neq \hat{\tau}$, then $\tau\hat{\tau}$ is a composition of 3-cycles.

To prove this, suppose $\tau = (a_1, a_2)$ and $\hat{\tau} = (a_3, a_4)$. There are two cases to consider:

(1) If $a_i \neq a_j$ for $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, then $(a_1, a_2)(a_3, a_4) = (a_1, a_3, a_2)(a_1, a_3, a_4)$.

(2) Otherwise $a_i = a_j$ for some $i \neq j$, and we can assume without loss of generality that $a_2 = a_4$. Then we have $(a_1, a_2)(a_3, a_2) = (a_1, a_2, a_3)$.

Thus $\tau\hat{\tau}$ is a composition of 3-cycles, completing the proof of Claim (b). □

c. Fix $r, s \in \{1, \ldots, n\}$ with $r \neq s$. If $n \geq 3$, then A_n is generated by the set of 3-cycles $\{(r, s, i) : 1 \leq i \leq n\}$.

Proof. After some manipulation, one can establish the identities:

(i) $(r,s,i)^2 = (s,r,i)$,
(ii) $(r,s,j)(r,s,i)^2 = (r,i,j)$,
(iii) $(r,s,j)^2(r,s,i) = (s,i,j)$,
(iv) $(r,s,i)^2(r,s,k)(r,s,j)^2(r,s,i) = (i,j,k)$.

Since every 3-cycle is of the form of one of those above, it follows that A_n is generated by the set of 3-cycles $\{(r,s,i) : 1 \leq i \leq n\}$. □

d. Suppose $n \geq 3$. Let $N \lhd A_n$ be a normal subgroup. If N contains a 3-cycle then $N = A_n$.

Proof. Suppose N contains a 3-cycle σ. Then $\sigma = (r,s,i)$ for some choice of $r,s,i \in \{1,\dots,n\}$. Observe (after some manipulation) that for any $j \neq i \in \{1,\dots,n\}$ we have

$$((r,s)(i,j))\,(r,s,i)^2\,((r,s)(i,j))^{-1} = (r,s,j).$$

The expression on the left in in N since it is a conjugate of an element of N. Thus N contains the set $\{(r,s,j) : 1 \leq j \leq n\}$. By virtue of part **c.**, it follows that $N = A_n$. □

e. Suppose $n \geq 5$. If $N \lhd A_n$ is a non-trivial normal subgroup, then N contains a 3-cycle.

Proof. We will do this in a case by case analysis. The first step is to show that if $N \lhd A_n$ is a non-trivial normal subgroup, then one of the following cases holds (note this is a slight simplification of the list given in the problem, but is essentially the same):

CASE I: There exists $\sigma \in N$ that can be written as a disjoint product of the form $\sigma = \mu(a_1,\dots,a_r)$ for some $r \geq 4$.

CASE II: There exists $\sigma \in N$ that can be written as a disjoint product of the form $\sigma = \mu(a_4,a_5,a_6)(a_1,a_2,a_3)$.

CASE III: There exists $\sigma \in N$ that can be written as a disjoint product of the form $\sigma = \mu(a_1,a_2,a_3)$, with μ a disjoint product of transpositions.

CASE IV: There exists $\sigma \in N$ that can be written as a disjoint product of the form $\sigma = \mu(a_3,a_4)(a_1,a_2)$, with μ a disjoint product of transpositions.

To see that one of these cases must hold, consider the fact that any non-trivial $\sigma \in S_n$ can be written as a product of disjoint cycles

$$\sigma = \sigma_1 \dots \sigma_m$$

for some $m \in \mathbb{N}$. Since disjoint cycles commute, we may reorder so that the length of the cycles is non-decreasing. The fact that one of the cases above must hold is then obvious.

Now we will show that in each case above, N contains a 3-cycle. For Case I, consider the expression $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$. This is in N since $(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is a conjugate of an element of N. On the other hand, after some algebra, one has

$$\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} = (a_1, a_3, a_r),$$

so that N contains a 3-cycle.

For Case II, consider the expression $\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1}$. Again this is clearly in N, and after some algebra one has

$$\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1} = (a_1, a_4, a_2, a_6, a_3).$$

Thus N contains a cycle of length five, and so by Case I, it also contains a cycle of length three.

For Case III, one has

$$\sigma^2 = (a_1, a_3, a_2)$$

using the fact that μ^2 is the identity (it is the product of disjoint transpositions). Thus N contains a 3-cycle.

Finally, for Case IV, consider $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$. Some algebra shows that this is equal to $(a_1, a_3)(a_2, a_4)$. We call this permutation α, which as above, is in N. Now let $\beta = (a_1, a_3, i)$ for some $i \in \{1, \ldots, n\} - \{a_1, \ldots, a_n\}$. Then

$$\beta^{-1}\alpha\beta\alpha = (a_1, a_3, i),$$

which again is in N for the same reason. Thus N contains a 3-cycle. □

Let us conclude by showing that A_n is simple for $n \geq 5$. Let $N \triangleleft A_n$ be a non-trivial normal subgroup of A_n. In the proof of part **e.** above, we showed that such a subgroup must contain a 3-cycle. In part **d.** we showed that if N contains a 3-cycle, then it is equal to A_n. This proves that the only normal subgroups of A_n are the trivial subgroup and A_n. Thus A_n is simple. □

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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