HOMEWORK EXAMPLE

JANE DOE

1. Exercises 14

Exercise 1 (# 14.30). Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for all $a \in G$.

Proof. Since H is normal in G, there is a group structure on G/H, the set of left cosets of H in G, given by the composition rule

$$(aH)(bH) = abH$$

for all $a, b \in G$. Note that the identity element of G/H is the coset H.

By the definition of the index, we have that $|G/H| = m$. From the corollary of Lagrange's Theorem, we know that for an $aH \in G/H$, the order of aH satisfies

$$|aH| \,|\, |G/H| = m.$$

In other words, for any $a \in G$, we have

$$(aH)^m = H$$

Using the composition rule, we also get

$$(aH)^m = a^m H.$$

We conclude that

$$a^m H = H$$

and thus that $a^m \in H$. □

University of Colorado at Boulder, Department of Mathematics, Campus Box 395, Boulder, CO 80309-0395, USA
E-mail address: doe@math.colorado.edu

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