HOMEWORK EXAMPLE

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1. Exercises 11

Exercise 1 (# 11.47). Let G be an abelian group. Let H be the subset of G consisting of the identity e together with all elements of order 2. Show that H is a subgroup of G.

Proof. Let G be an abelian group. Let H be the subset of G consisting of the identity e together with all elements of order 2.

To show that H is a subgroup, it suffices to show that for all $a, b \in H$, one has $ab^{-1} \in H$. So let $a, b \in H$. Then

$$(ab^{-1})(ab^{-1}) = ab^{-1}ab^{-1} = aab^{-1}b^{-1} = aa(bb)^{-1} = ee = e.$$

Thus $ab^{-1} \in H$. □

2. Exercises 13

Exercise 2 (# 13.47). Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.

Proof. Let $\phi : G \to G'$ be a group homomorphism where $|G|$ is a prime. Let $e' \in G'$ be the identity element. The problem asks us to show that either $\phi(g) = e'$ for all $g \in G$, or, ϕ is injective.

To prove this, let us consider $\ker \phi$. The kernel of a homomorphism is a subgroup of G, and, since $|G|$ is finite, $|\ker \phi|$ divides $|G|$ (Theorem of Lagrange). By virtue of the fact that $|G|$ is prime, it follows that either $|\ker \phi| = 1$ or $|\ker \phi| = |G|$. That is, either $\ker \phi = \{e\}$, where e is the identity element of G, or $\ker \phi = G$.

In the former case, ϕ is injective (a homomorphism is injective if and only if the kernel is trivial). In the latter case, $\phi(g) = e'$ for all $g \in G$, from the definition of the kernel. □

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