CHAPTER 6

# A brief introduction to linear algebra

## 1. Vector spaces and linear maps

In what follows, fix $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. More generally, $K$ can be any field.

**1.1. Vector spaces.** Motivated by our intuition of adding and scaling vectors in the plane (see Figure 1), we make the following definition:

**Definition 6.1.1.** *A $K$-**vector space** consists of a triple $(V, +, \cdot)$, where $V$ is a set, and $+ : V \times V \to V$ and $\cdot : K \times V \to V$ are maps, satisfying the following properties:*

(1) *(Group laws)*
   (a) *(Additive identity) There exists an element $\mathscr{O} \in V$ such that for all $v \in V$, $v + \mathscr{O} = v$;*
   (b) *(Additive inverse) For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathscr{O}$;*
   (c) *(Associativity of addition) For all $v_1, v_2, v_3 \in V$,*
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$
(2) *(Abelian property)*
   (a) *(Commutativity of addition) For all $v_1, v_2 \in V$,*
$$v_1 + v_2 = v_2 + v_1;$$
(3) *(Module conditions)*
   (a) *For all $\lambda \in K$ and all $v_1, v_2 \in V$,*
$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2);$$
   (b) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v);$$
   (c) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v);$$
   (d) *For all $v \in V$,*
$$1 \cdot v = v.$$

*In the above, for all $\lambda \in K$ and all $v, v_1, v_2 \in V$ we have denoted $+(v_1, v_2)$ by $v_1 + v_2$ and $\cdot(\lambda, v)$ by $\lambda \cdot v$.*

In addition, for brevity, we will often write $\lambda v$ for $\lambda \cdot v$.

EXAMPLE 6.1.2 (The vector space $K^n$). By definition,
$$K^n = \{(x_1, \ldots, x_n) \ : \ x_i \in K, \ 1 \le i \le n\}.$$
The map $+ : K^n \times K^n \to K^n$ is defined by the rule
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

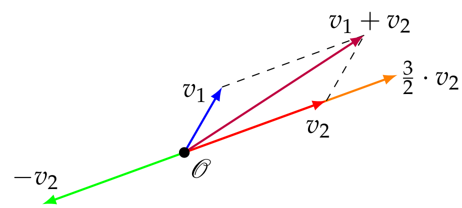


FIGURE 1. Adding and scaling vectors in the plane

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K^n$. The map $\cdot : K \times K^n \to K^n$ is defined by the rule

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

for all $\lambda \in K$ and $(x_1, \ldots, x_n) \in K^n$.

**Exercise 6.1.3.** *Show that $(K^n, +, \cdot)$, defined in the example above, is a K-vector space.*

**Exercise 6.1.4** (Cancelation rule)**.** *Let $(V, +, \cdot)$ be a K-vector space. Show that if we have $v_1, v_2, w \in V$, then*

$$v_1 + w = v_2 + w \iff v_1 = v_2.$$

**Exercise 6.1.5** (Unique additive identity)**.** *Let $(V, +, \cdot)$ be a K-vector space. Fix an element $\mathscr{O} \in V$ such that for all $v \in V$, we have $v + \mathscr{O} = v$. Show that if $w \in V$ satisfies $v' + w = v'$ for all $v' \in V$, then $w = \mathscr{O}$.*

**Exercise 6.1.6** (Unique additive inverse)**.** *Let $(V, +, \cdot)$ be a K-vector space. Let $v \in V$. Fix an element $-v \in V$ such that $v + (-v) = \mathscr{O}$. Suppose that there is $w \in V$ such that $v + w = \mathscr{O}$. Show that $w = -v$.*

**Exercise 6.1.7.** *Let $(V, +, \cdot)$ be a K-vector space. Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in K$.*

(1) $0v = \mathscr{O}$.
(2) $\lambda \mathscr{O} = \mathscr{O}$.
(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.
(4) *If* $\lambda v = \mathscr{O}$, *then either* $\lambda = 0$ *or* $v = \mathscr{O}$.
(5) *If* $\lambda v_1 = \lambda v_2$, *then either* $\lambda = 0$ *or* $v_1 = v_2$.
(6) *If* $\lambda_1 v = \lambda_2 v$, *then either* $\lambda_1 = \lambda_2$ *or* $v = \mathscr{O}$.
(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.
(8) $v + v = 2v$, $v + v + v = 3v$, *and in general* $\sum_{i=1}^{n} v = nv$.

**Exercise 6.1.8.** *Consider the set of maps from a set S to K. Let us denote this set by* $\mathrm{Map}(S, K)$. *Define addition and multiplication maps*

$$+ : \mathrm{Map}(S, K) \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

*and*

$$\cdot : K \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

*in the following way. For all $f, g \in \mathrm{Map}(S, K)$, set $f + g$ to be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For all $\lambda \in K$ and all $f \in \mathrm{Map}(S, K)$, set $\lambda \cdot f$ to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ for all $x \in S$.* Show that if $S \neq \emptyset$ then $(\mathrm{Map}(S, K), +, \cdot)$ is a $K$-vector space.

## 2. Sub-vector spaces

**Definition 6.2.9** (sub-$K$-vector space)**.** *Let* $(V, +, \cdot)$ *be a* $K$*-vector space. A* ***sub-$K$-vector space*** *of* $(V, +, \cdot)$ *is a* $K$*-vector space* $(V', +', \cdot')$ *such that* $V' \subseteq V$ *and such that for all* $v', v'_1, v'_2 \in V'$ *and all* $\lambda \in K$,

$$v'_1 +' v'_2 = v'_1 + v'_2 \quad \text{and} \quad \lambda \cdot' v' = \lambda \cdot v'.$$

*We will write* $(V', +', \cdot') \subseteq (V, +, \cdot)$.

**Definition 6.2.10.** *If* $(V, +, \cdot)$ *is a* $K$*-vector space, and* $V' \subseteq V$ *is a subset, we say that* $V'$ *is* ***closed under*** $+$ ***(resp. closed under*** $\cdot$***)*** *if for all* $v'_1, v'_2 \in V'$ *(resp. for all* $\lambda \in K$ *and all* $v' \in V'$*) we have* $v'_1 + v'_2 \in V'$ *(resp.* $\lambda \cdot v' \in V'$*). In this case, we define*

$$+|_{V'} : V' \times V' \to V'$$

*(resp.* $\cdot|_{V'} : K \times V' \to V'$*) to be the map given by* $v'_1 +|_{V'} v'_2 = v'_1 + v'_2$ *(resp.* $\lambda \cdot |_{V'} v' = \lambda \cdot v'$*), for all* $v'_1, v'_2 \in V'$ *(resp. for all* $\lambda \in K$ *and all* $v' \in V'$*).*

REMARK 6.2.11. Note that if $(V', +', \cdot')$ is a sub-$K$-vector space of $(V, +, \cdot)$, then $V'$ is closed under $+$ and $\cdot$.

**Exercise 6.2.12.** *Show that if a non-empty subset* $V' \subseteq V$ *is closed under* $+$ *and* $\cdot$*, then* $(V', +|_{V'}, \cdot|_{V'})$ *is a sub-*$K$*-vector space of* $(V, +, \cdot)$*.*

**Exercise 6.2.13.** *Show that if* $(V', +', \cdot')$ *is a sub-*$K$*-vector space of a* $K$*-vector space* $(V, +, \cdot)$*, then the additive identity element* $\mathscr{O}' \in V'$ *is equal to the additive identity element* $\mathscr{O} \in V$*.*

**Exercise 6.2.14.** *Recall the* $\mathbb{R}$*-vector space* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$ *from Exercise 6.1.8. In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ *listed below are closed under* $+$ *and* $\cdot$*, and so define sub-*$\mathbb{R}$*-vector spaces of* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$*.*

(1) *The set of all polynomial functions.*
(2) *The set of all polynomial functions of degree less than* $n$*.*
(3) *The set of all functions that are continuos on an interval* $(a, b) \subseteq \mathbb{R}$*.*
(4) *The set of all functions differentiable at a point* $a \in \mathbb{R}$*.*
(5) *The set of all functions differentiable on an interval* $(a, b) \subseteq \mathbb{R}$*.*
(6) *The set of all functions with* $f(1) = 0$*.*
(7) *The set of all solutions to the differential equation* $f'' + af' + bf = 0$ *for some* $a, b \in \mathbb{R}$*.*

**Exercise 6.2.15.** *In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ *listed below are NOT closed under* $+$ *and* $\cdot$*, and so do not define sub-*$\mathbb{R}$*-vector spaces of* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$*.*

(1) *Fix* $a \in \mathbb{R}$ *with* $a \neq 0$*. The set of all functions with* $f(1) = a$*.*
(2) *The set of all solutions to the differential equation* $f'' + af' + bf = c$ *for some* $a, b, c \in \mathbb{R}$ *with* $c \neq 0$*.*

## 3. Linear maps

**Definition 6.3.16** (Linear map)**.** *Let* $(V, +, \cdot)$ *and* $(V', +', \cdot')$ *be* $K$*-vector spaces. A* ***linear map*** $F : (V, +, \cdot) \to (V', +', \cdot')$ *is a map of sets*

$$f : V \to V'$$

*such that for all* $\lambda \in K$ *and* $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \text{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. The $K$-vector space $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and the $K$-vector space $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of $f$.

**Exercise 6.3.17.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that the image of* $f$ *is closed under* $+', \cdot'$*, and so defines a sub-K-vector space of the target* $(V', +', \cdot')$.

**Exercise 6.3.18.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that* $f(\mathscr{O}) = \mathscr{O}'$.

**Exercise 6.3.19.** *Show that the following maps of sets define linear maps of the K-vector spaces.*

(1) *Let* $(V, +, \cdot)$ *be a K-vector space. Show that the identity map* $f : V \to V$*, given by* $f(v) = v$ *for all* $v \in V$*, is a linear map. This linear map will frequently be denoted by* $\mathrm{Id}_V$.
(2) *Let* $(V, +, \cdot)$ *and* $(V', +', \cdot')$ *be K-vector spaces. Show that the zero map* $f : V \to V'$*, given by* $f(v) = \mathscr{O}'$ *for all* $v \in V$*, is a linear map.*
(3) *Let* $(V, +, \cdot)$ *be a K-vector space and let* $\alpha \in K$*. Show that the multiplication map* $f : V \to V$ *given by* $f(v) = \alpha \cdot v$ *for all* $v \in V$ *is a linear map. This linear map will frequently be denoted by* $\alpha \, \mathrm{Id}_V$.
(4) *Let* $a_{ij} \in K$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*. Show that the map* $f : K^n \to K^m$ *given by*

$$f(x_1, \ldots, x_n) = \left( \sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right)$$

*is a linear map.*
(5) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all differentiable real functions* $g : \mathbb{R} \to \mathbb{R}$*. Let* $(V', +', \cdot')$ *be the* $\mathbb{R}$*-vector space of all real functions* $g : \mathbb{R} \to \mathbb{R}$*. Show that the map* $f : (V, +, \cdot) \to (V', +', \cdot')$ *that sends a differentiable function* $g$ *to its derivative* $g'$ *is a linear map.*
(6) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all continuous real functions* $f : \mathbb{R} \to \mathbb{R}$*. Show that the map* $f : (V, +, \cdot) \to (V, +, \cdot)$ *that sends a function* $g \in V$ *to the function* $f(g) \in V$ *determined by*

$$f(g)(x) := \int_a^x g(t) dt \quad \text{for all } x \in \mathbb{R}$$

*is a linear map. Make sure to show that* $f(g) \in V$ *for all* $g \in V$.

**Definition 6.3.20** (Kernel)**.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. The* ***kernel of*** $f$ *(or Null space of* $f$*), denoted* $\ker(f)$ *(or* $\mathrm{Null}(f)$*), is the set*

$$\ker(f) := f^{-1}(\mathscr{O}') = \{v \in V : f(v) = \mathscr{O}'\}.$$

**Exercise 6.3.21.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that* $\ker(f)$ *is a sub-K-vector space of* $(V, +, \cdot)$.

**Exercise 6.3.22.** *Find the kernel of each of the linear maps listed below (see Problem 6.3.19).*

(1) *The linear map* $\mathrm{Id}_V$.
(2) *The zero map* $V \to V'$.
(3) *The linear map* $\alpha\,\mathrm{Id}_V$.
(4) *Let* $a_{ij} \in K$ *for* $1 \le i \le m$ *and* $1 \le j \le n$. *The linear map* $f : K^n \to K^m$ *defined by*

$$f(x_1, \ldots, x_n) = \left( \sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{ij}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j \right).$$

(5) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all differentiable real functions* $g : \mathbb{R} \to \mathbb{R}$. *Let* $(V', +', \cdot')$ *be the* $\mathbb{R}$*-vector space of all real functions* $g : \mathbb{R} \to \mathbb{R}$. *The linear map* $f : (V, +, \cdot) \to (V', +', \cdot')$ *that sends a differentiable function* $g$ *to its derivative* $g'$.
(6) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all continous real functions* $g : \mathbb{R} \to \mathbb{R}$. *Let* $a \in \mathbb{R}$. *The linear map* $f : (V, +, \cdot) \to (V, +, \cdot)$ *that sends a function* $g \in V$ *to the function* $f(g) \in V$ *determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}.$$

**Exercise 6.3.23.** *Show that the composition of linear maps is a linear map.*

**Definition 6.3.24** (Isomorphism)**.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of* $K$*-vector spaces. We say that* $f$ *is an isomorphism of* $K$*-vector spaces if there is a linear map* $g : (V', +', \cdot') \to (V, +, \cdot)$ *of* $K$*-vector spaces such that*

$$g \circ f = \mathrm{Id}_{(V,+,\cdot)} \quad \text{and} \quad f \circ g = \mathrm{Id}_{(V',+',\cdot')}.$$

**Exercise 6.3.25.** *Show that a linear map is an isomorphism if and only if it is bijective.*

## 4. Bases and dimension

**4.1. Linear maps determined by elements of a vector space.** The basic example we are interested in is the following. Let $V$ be a $K$-vector space. We fix

$$\mathbf{v} = (v_1, \ldots, v_n) \in V^n.$$

From this we obtain a map

$$L_{\mathbf{v}} : K^n \to V$$
$$(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i v_i.$$

**Exercise 6.4.26.** *Show that* $L_{\mathbf{v}}$ *is a linear map.*

**4.2. Span, linear independence, and bases.** For every permutation $\sigma \in \Sigma_n$, the symmetric group on $n$-letters, we set

$$\mathbf{v}^\sigma := (v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

**Definition 6.4.27.** *Let* $V$ *be a* $K$*-vector space, and let* $v_1, \ldots, v_n \in V$. *Set* $\mathbf{v} = (v_1, \ldots, v_n)$. *We say:*

(1) *The elements* $v_1, \ldots, v_n$ ***span*** $V$ *(or generate* $V$*) if for every* $\sigma \in \Sigma_n$*, the linear map* $L_{\mathbf{v}^\sigma}$ *is surjective.*

(2) *The elements $v_1, \dots, v_n$ are **linearly independent** if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \dots, v_n$ are a **basis for** $V$ if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

**Exercise 6.4.28.** *Let $V$ be a $K$-vector space, and let $v_1, \dots, v_n \in V$. Set $\mathbf{v} = (v_1, \dots, v_n)$.*
(1) *The elements $v_1, \dots, v_n$ **span** $V$ (or generate $V$) if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is surjective.*
(2) *The elements $v_1, \dots, v_n$ are **linearly independent** if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \dots, v_n$ are a **basis for** $V$ if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

**Exercise 6.4.29.** *Let $V$ be a $K$-vector space, and let $v_1, \dots, v_n \in V$.*
(1) *The elements $v_1, \dots, v_n$ **span** $V$ (or generate $V$) if for any $v \in V$, there exists $(a_1, \dots, a_n) \in K^n$ such that $\sum_{i=1}^n a_i v_i = v$.*
(2) *The elements $v_1, \dots, v_n$ are **linearly independent** if whenever $(a_1, \dots, a_n) \in K^n$ and $\sum_{i=1}^n a_i v_i = 0$, we have $(a_1, \dots, a_n) = 0$.*
(3) *The elements $v_1, \dots, v_n$ are a **basis for** $V$ if they span $V$ and are linearly independent.*

**4.3. Dimension.** We start with the following motivational exercise:

**Exercise 6.4.30.** *If $K^n \cong K^m$, then $n = m$.*

**Definition 6.4.31.** *A $K$-vector space $V$ is said to be of dimension $n$ if there is an isomorphism $V \cong K^n$.*

**Exercise 6.4.32.** *Show that a $K$-vector space $V$ has dimension $n$ if and only if it has a basis consisting of $n$ elements.*

## 5. Direct products of vector spaces

EXAMPLE 6.5.33. Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are $K$-vector spaces. There is a $K$-vector space

$$(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$$

where $V_1 \times V_2$ is the product of the sets $V_1$ and $V_2$, where

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

**Exercise 6.5.34.** *Show that the triple $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$ in the example above is a $K$-vector space.*

**Definition 6.5.35 (Direct product).** *Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are $K$-vector spaces. We define the* direct product *of $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$, written $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2)$, to be the $K$-vector space $(V_1 \times V_2, +, \cdot)$ defined above.*

**Exercise 6.5.36.** *Let $V_1$ and $V_2$ be K-vector spaces. Show the following:*

(1) *There is an injective linear map $i_1 : V_1 \to V_1 \times V_2$ given by $v_1 \mapsto (v_1, \mathscr{O}_{V_2})$, and a surjective linear map $p_1 : V_1 \times V_2 \to V_1$ given by $(v_1, v_2) \mapsto v_1$.*
(2) *There is an injective linear map $i_2 : V_1 \to V_1 \times V_2$ given by $v_2 \mapsto (\mathscr{O}_{V_1}, v_2)$, and a surjective linear map $p_2 : V_1 \times V_2 \to V_2$ given by $(v_1, v_2) \mapsto v_2$.*

## 6. Quotient vector spaces

Suppose that $(V, +, \cdot)$ is a $K$-vector space, and $W \subseteq V$ is a sub-$K$-vector space. Define an equivalence relation on $V$ by the rule

$$v_1 \sim v_2 \iff v_1 - v_2 \in W.$$

**Exercise 6.6.37.** *Show that this defines an equivalence relation on $V$.*

Let $V/W$ be the set of equivalence classes, and let

$$\pi : V \longrightarrow V/W$$

be the quotient map of sets. For any element $\nu \in V/W$, there is an element $v \in V$ such that $\nu = [v]$, where $[v]$ is the equivalence class of $v$.

**Exercise 6.6.38.** *Let $V$ be a K-vector space and suppose that $W \subseteq V$ is a sub-K-vector space.*

(1) *Suppose that $[v_1], [v_2] \in V/W$. Show that the rule*
$$[v_1] + [v_2] = [v_1 + v_2]$$
*defines a map*
$$+ : V/W \times V/W \to V/W.$$
(2) *Suppose that $\lambda \in K$ and $[v] \in V/W$. Show that the rule*
$$\lambda \cdot [v] = [\lambda \cdot v]$$
*defines a map*
$$\cdot : K \times V/W \to V/W.$$
(3) *Show that $V/W$ is a K-vector space with $+$ and $\cdot$ defined as above.*
(4) *Show that $\pi : V \to V/W$ is a surjective linear map with kernel $W$.*

**Definition 6.6.39** (Quotient $K$-vector space)**.** *Let $V$ be a K-vector space and let $W \subseteq V$ be a sub-K-vector space. The quotient (K-vector space) of $V$ by $W$ is the K-vector space $V/W$ constructed above.*

**Exercise 6.6.40.** *Suppose that $\phi : V \twoheadrightarrow V'$ is a surjective linear map of K-vector spaces.*

(1) *Show that $V' \cong V/\ker \phi$.*
(2) *If $V'$ is finite dimensional, show that $V \cong (\ker \phi) \times V'$.*
(3) *If $V$ and $V'$ are finite dimensional, show that $\dim V = \dim V' + \dim(\ker \phi)$.*

## 7. Further exercises

**Exercise 6.7.41.** *Find an example of a triple $(V, +, \cdot)$ satisfying all of the conditions of the definition of a K-vector space, except for condition (3)(d).*

**Exercise 6.7.42.** *Suppose that* $L : K^n \to K^m$ *is a linear map. For* $j = 1, \ldots, n$ *define* $e_j = (0, \ldots, 1, \ldots, 0) \in K^n$ *to be the element with all entries* 0 *except for the* $j$*-th place, which is* 1*. Similarly, for* $i = 1, \ldots, m$ *define* $f_i^\vee : K^m \to K$ *to be the linear map defined by* $(y_1, \ldots, y_m) \mapsto y_i$*. Show that* $L$ *is the same as the linear map defined in Example 6.3.19(4) with the matrix* $A \in \mathrm{M}_{m \times n}(K)$ *defined by* $A_{ij} = a_{ij} = f_i^\vee(L(e_j))$*.*