FINAL EXAM
LINEAR ALGEBRA

MATH 2130
SUMMER 2019

Friday July 5, 2019
9:15 AM – 10:50 AM

Name

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	9	
10	10	10	10	10	10	10	10	10	total

Date: July 3, 2019.

1
10 points

1. Find the determinant of each of the following matrices.

1.(a). $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

1.(b). $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

SOLUTION

(a) We have $\boxed{\det A = -1}$ The fastest way to see this may be to expand off of the third column; however, to use the standard method, we have

$$\det A = (4)[(-2)(0) - (0)(1)] - (-1)[(-1)(0) - (0)(0)] + (1)[(-1)(1) - (-2)(0)] = -1.$$

(b) We have $\boxed{\det B = -2}$ We use row operations:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{vmatrix}$$

$$
\begin{aligned}
&= (-1)^3 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \\
&= -2
\end{aligned}
$$

2
10 points

2. Let $V = \mathbb{R}[x]$ be the vector space of real polynomial functions. Let

$$D : V \to V$$

be the derivative map; i.e. $D(p) = p'$ for all $p \in V$. Let

$$E : V \to V$$

be the integration map that sends a polynomial p to the polynomial q given by $q(x) = \int_0^x p(t)dt$, for all $x \in \mathbb{R}$. It is a fact that D and E are linear maps.

2.(a). *Show that D is surjective, but not injective.*

2.(b). *Show that E is injective, but not surjective.*

SOLUTION

We will show first that $(ED)(p) = p - p(0)$, and $(DE)(p) = p$ (we will only use the latter). We have

$$\begin{aligned}
((ED)(p))(x) &= (E(D(p)))(x) \\
&= \int_0^x D(p)(t)dt = \int_0^x p'(t)dt \\
&= p(x) - p(0) \\
((DE)(p))(x) &= (D(E(p)))(x) \\
&= \frac{d}{dx}(E(p)(x)) \\
&= \frac{d}{dx}\int_0^x p(t)dt \\
&= p(x).
\end{aligned}$$

(a) Since $DE : V \to V$ is the identity, and in particular is surjective, we must have that D is surjective (more directly, we can prove the surjectivity of D by observing that every polynomial has an anti-derivative that is a polynomial). On the other hand, D is not injective, since $D(p) = 0$ for every constant polynomial p.

(b) Since $DE : V \to V$ is the identity, and in particular is injective, we must have that E is injective (more directly, we can prove the injectivity of E by observing that every polynomial has an anti-derivative that is a polynomial of degree at least 1, and arguing from there). On the other hand, E is not surjective, since, for instance, there is no polynomial p such that $\int_0^x p(t)dt = 1$ for all x (for instance plug in $x = 0$ and we get $\int_0^0 p(t)dt = 0 \neq 1$).

3
10 points

3. Suppose we have a two state Markov chain with stochastic matrix

$$P = \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix}$$

Given the probability vector $v = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$, *find* $\lim_{n\to\infty} P^n v$.

SOLUTION

The solution is $\boxed{\lim_{n\to\infty} P^n v = \begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}}$

Indeed, since P is a stochastic matrix with positive entries, given any probability vector v, we have $\lim_{n\to\infty} P^n v$ is the unique probability vector that is an eigenvector with eigenvalue 1. So to obtain the solution, we first find an eigenvector with eigenvalue 1. For this, we are trying to find the kernel of

$$1 \cdot I - P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.5 \\ -0.9 & 0.5 \end{pmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & -5/9 \\ 0 & 0 \end{pmatrix}$$

and then we augment the matrix:

$$\begin{pmatrix} 1 & -5/9 \\ 0 & -1 \end{pmatrix}$$

This tells us that $\begin{pmatrix} -5/9 \\ -1 \end{pmatrix}$, or more conveniently, $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$, gives a basis for the $\lambda = 1$ eigenspace. The corresponding probability vector is $\begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}$.

4. Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

4
10 points

4.(a). *Find the characteristic polynomial $p_A(t)$ of A.*

4.(b). *Find the eigenvalues of A.*

4.(c). *Find an orthonormal basis for each eigenspace of A in $\mathbb{R}^3$.*

4.(d). *Is A diagonalizable? If so, find a matrix $S \in M_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

4.(e). *Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix $U \in M_{3\times 3}(\mathbb{R})$ so that $U^T AU$ is diagonal. If not, explain.*

SOLUTION

(a) We have

$$\begin{aligned} p_A(t) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-3 & 1 \\ -2 & -1 & t-3 \end{vmatrix} \\ &= (t-2)[(t-3)^2 - (1)(-1)] - (1)[0-(1)(-2)] + (-1)[0-(t-3)(-2)] \\ &= (t-2)[t^2-6t+10] - 2 + \underbrace{(t-3)(-2)}_{-2t+6} \\ &= (t^3 - 6t^t + 10t - 2t^2 + 12t - 20) - 2 + (6-2t) \\ &= t^3 - 8t^2 + 20t - 16. \end{aligned}$$

In other words, the solution is:

$$\boxed{p_A(t) = t^3 - 8t^2 + 20t - 16.}$$

As a quick partial check of the solution, observe that

$$\begin{aligned} \operatorname{tr}(A) &= 8 \\ \det A &= \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2(6+2) = 16. \end{aligned}$$

confirming the computation of the coefficients of t^2 and t^0.

(b) One can easily check that

$$p_A(2) = 2^3 - 8 \cdot 2^2 + 20 \cdot 2 - 16 = 8 - 32 + 40 - 16 = 48 - 48 = 0.$$

Thus we have

$$p_A(t) = (t-2)(t^2 - 6t + 8) = (t-2)(t-2)(t-4).$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 4.}$$

(c) To find the $\lambda = 2$ eigenspace E_2, we compute

$$E_2 := \ker(2I - A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We add rows, and get the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ \color{red}{0} & \color{red}{0} & \color{green}{-1} \end{pmatrix}$$

Thus we have

$$E_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ \color{green}{-1} \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Now we compute the $\lambda = 4$ eigenspace E_4. We have

$$E_4 = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ \color{red}{0} & \color{red}{0} & \color{green}{-1} \end{pmatrix}$$

Thus we have

$$E_4 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Thus the solution to the problem is:

The eigenspaces for A are E_2 and E_4, and we have that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is a basis for E_2 and $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ is a basis for E_4.

It follows that

- $\boxed{\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$ *is an orthonormal basis for E_2, and,*
- $\boxed{\frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$ *is an orthonormal basis for E_4.*

Note that we can easily double check that the given basis elements are eigenvectors.

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+1-1 \\ -3+1 \\ 2-1-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-1-1 \\ 3+1 \\ -2+1-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$$

(d) $\boxed{\text{No.}}$ A is not diagonalizable since $\mathbb{R}^3$ does not admit a basis of eigenvectors for A.

(e) $\boxed{\text{No.}}$ A is not diagonalizable with orthogonal matrices either, since it is not even diagonalizable.

5. Consider the following matrix:

5
10 points

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

5.(a). *What is the sum of the roots of the characteristic polynomial of B?*

5.(b). *What is the product of the roots of the characteristic polynomial of B?*

5.(c). *Are the roots of the characteristic polynomial of B real?*

SOLUTION

(a) The sum of the roots of the characteristic polynomial of B is equal to the trace of B. So we have

$$\operatorname{tr} B = 0 + 0 + 2 + 1 + 8 + 0 = 11.$$

So the answer is $\boxed{11}$.

(b) The product of the roots of the characteristic polynomial of B is equal to the determinant of B (since it is a 6×6 matrix). Since B is block-upper-triangular, we could compute the determinant that way; but the fourth and fifth rows are linearly dependent, so the determinant is 0. Thus the answer is $\boxed{0}$.

(c) $\boxed{\text{No.}}$ We have

$$P_B(t) = \left|\begin{array}{cc|cccc} t & -1 & 0 & -2 & 1 & 0 \\ 1 & t & -2 & -1 & -1 & -1 \\ \hline 0 & 0 & t-2 & 0 & 2 & 0 \\ 0 & 0 & 0 & t-1 & -4 & -3 \\ 0 & 0 & 0 & -2 & t-8 & -6 \\ 0 & 0 & 0 & -3 & 3 & t \end{array}\right| = (t^2+1)p(t)$$

where $p(t)$ is the determinant of the lower right block in the matrix above. Thus $\pm i$ are roots of $p_B(t)$.

6
10 points

6. Consider the two dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

6.(a). *Is the origin an attractor, repeller, or saddle point?*

6.(b). *Find the directions of greatest attraction or repulsion.*

SOLUTION

(a) The origin is a $\boxed{\text{saddle point.}}$

To see this, we compute that the characteristic polynomial is

$$p_A(t) = \det \begin{pmatrix} t-1.7 & -0.3 \\ -1.2 & t-0.8 \end{pmatrix} = (t^2 - 2.5t + 1.36) - (.36) = t^2 - 2.5t + 1$$

$$= (t-2)(t-\frac{1}{2})$$

Thus the eigenvalues are $\lambda = \frac{1}{2}, 2$. Since $0 < \frac{1}{2} < 1$ and $1 < 2$, we see that the origin is a saddle point.

(b) We have that the line spanned by $\boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix}}$ is the direction of greatest attraction, and the line spanned by $\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is the direction of greatest repulsion.

To deduce this, we find the eigenspaces. We start with the $\lambda = \frac{1}{2}$-eigenspace, $E_{1/2}$, which is the kernel of $\frac{1}{2}I - A$:

$$\frac{1}{2}I - A = \begin{pmatrix} -1.2 & -0.3 \\ -1.2 & -0.3 \end{pmatrix} \mapsto \begin{pmatrix} 12 & 3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & -1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is a basis for the $\frac{1}{2}$-eigenspace $E_{1/2}$.

We now compute the $\lambda = 2$-eigenspace, E_2, which is the kernel of $2I - A$:

$$2I - A = \begin{pmatrix} 0.3 & -0.3 \\ -1.2 & 1.2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the 2-eigenspace E_2.

In conclusion, the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion.

7. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

7
10 points

Find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ *spanned by* $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, *and* $\mathbf{x}_4$.

SOLUTION

An orthonormal basis is given by

$$\boxed{\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}}$$

Note, that before we get started, it is a good idea to check whether the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are linearly dependent, since otherwise, we could take the standard basis in $\mathbb{R}^4$ as a solution, and be done immediately. We can check linear dependence via row reduction of the associated 4×4 matrix, or by simply noting in this case that $\mathbf{x}_4 = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3$. Thus $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are linearly dependent, so we cannot simply take the standard basis in $\mathbb{R}^4$, and will have to use Gram–Schmidt instead. As a small benefit, we have found that the span of $\mathbf{x}_1, \ldots, \mathbf{x}_4$ is the same as the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and so we will simply apply Gram–Schmidt to $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

We start by finding an orthogonal basis. We have

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ -1/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

For simplicity, we will take

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

We have

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_2}{\mathbf{y}_2 \cdot \mathbf{y}_2}\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{15}\begin{bmatrix} 0 \\ 0 \\ 15 \\ 15 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} 2 \\ -4 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 \\ -9 \\ 9 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Again for simplicity we take

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Therefore, an orthogonal basis for the span of $\mathbf{x}_1, \ldots, \mathbf{x}_4$ is given by

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Consequently, an orthonormal basis is given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

Note that as a quick check, we have $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_2 = -\mathbf{x}_1 + 3\mathbf{x}_2$, $\mathbf{y}_3 = \mathbf{x}_1 + 2\mathbf{x}_2 - 5\mathbf{x}_3$. So $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are all in the span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. And one can check quickly that they are orthogonal.

8. *Find the equation $y = \beta_0 + \beta_1 x$ of the line that best fits the given data points, as a least squares model:*

8
10 points

$$\begin{bmatrix} x \\ y \end{bmatrix}: \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

SOLUTION

The best fit line is

$$\boxed{y = \frac{4}{5} + \frac{2}{5}x}$$

To find this, we have the matrices:

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \widetilde{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix},$$

The best fit line is given by β satisfying

$$\widetilde{\mathbf{x}}^T \widetilde{\mathbf{x}} \beta = \widetilde{\mathbf{x}}^T \mathbf{y}$$

or, since $\ker \widetilde{\mathbf{x}} = 0$,

$$\beta = (\widetilde{\mathbf{x}}^T \widetilde{\mathbf{x}})^{-1} \widetilde{\mathbf{x}}^T \mathbf{y}.$$

Using this latter formulation, we have

$$\begin{aligned} \beta &= \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ &= \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ &= \frac{1}{20} \begin{bmatrix} 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix} \end{aligned}$$

9
10 points

9. **TRUE** or **FALSE**. You do **not** need to justify your answer.

9.(a). Suppose A and B are invertible $n \times n$ matrices, and that $AB = BA$. Then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

T F | TRUE: $(AB)^{-1} = B^{-1}A^{-1}$ and $(BA)^{-1} = A^{-1}B^{-1}$.

9.(b). Let $f : V \to V$ be a linear map of a vector space to itself. If f is surjective, then f is an isomorphism.

T F | FALSE: We have seen examples where this fails. If V were assumed to be finite dimensional, however, then this statement would be true.

9.(c). Suppose that P is an $n \times n$ matrix with positive entries, such that the column sums are equal to 1. Then $\lim_{n\to\infty} P^n$ exists.

T F | TRUE: We have seen this in class.

9.(d). Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \mathrm{Im}(T)$.

T F | FALSE: Take $V = \mathbb{R}$ and $V' = 0$. (The Rank–Nullity Theorem states that $\dim V = \dim \ker(T) + \dim \mathrm{Im}(T)$.)

9.(e). If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors.

T F | TRUE: We have seen this in class.

9.(f). If v is an eigenvector for an $n \times n$ matrix A with eigenvalue λ, and $r \neq 0$ is a real number, then rv is an eigenvector for A with eigenvalue λ.

T F | TRUE: $A(rv) = rAv = r\lambda v = \lambda(rv)$.

9.(g). Suppose that $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ is symmetric, and let $v_1, v_2 \in \mathbb{R}^n$ be eigenvectors with corresponding eigenvalues λ_1, λ_2. If $\lambda_1 \neq \lambda_2$, then v_1 is orthogonal to v_2.

T F | TRUE: $\lambda_1(v_1.v_2) = Av_1.v_2 = v_1.A^Tv_2 = v_1.Av_2 = \lambda_2(v_1.v_2)$; since $\lambda_1 \neq \lambda_2$, we must have $v_1.v_2 = 0$.

9.(h). Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

T F | FALSE: The matrix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $M^2 = 0$, but M is not diagonalizable. (Note more generally that if $M = S^{-1}DS$ for a diagonal matrix D, then $0 = M^n = S^{-1}D^nS$ if and only if $D = 0$ (and hence $M = 0$), since S and S^{-1} induce isomorphisms.)

9.(i). For an $n \times n$ matrix A, if $\det(\mathrm{cof}\, A) = 0$, then $\det A = 0$.

T F | TRUE: We know that $A(\operatorname{cof} A)^T = (\det A)I$, so that $0 = (\det A)(\det(\operatorname{cof} A)) = (\det A)(\det((\operatorname{cof} A)^T)) = (\det A)^n$.

9.(j). Let $v, w \in \mathbb{R}^n$. If θ is the angle between v and w, then $\cos\theta = \dfrac{v.w}{||v||||w||}$.

T F | TRUE: This was our definition of the angle between vectors.