## FINAL EXAM LINEAR ALGEBRA

## MATH 2130 SUMMER 2019

Friday July 5, 2019 9:15 AM – 10:50 AM

Name

**PRACTICE EXAM** 

Please answer all of the questions, and show your work. You must explain your answers to get credit. You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	9	
10	10	10	10	10	10	10	10	10	total

*Date*: July 3, 2019.

**1.** Find the determinant of each of the following matrices.

**1.(a).** 
$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
  
**1.(b).** 
$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$$

1 10 points **2.** Let  $V = \mathbb{R}[x]$  be the vector space of real polynomial functions. Let  $D: V \to V$ be the derivative map; i.e. D(p) = p' for all  $p \in V$ . Let  $E: V \to V$  2 10 points

be the integration map that sends a polynomial p to the polynomial q given by  $q(x) = \int_0^x p(t)dt$ , for all  $x \in \mathbb{R}$ . It is a fact that D and E are linear maps.

**2.(a).** Show that *D* is surjective, but not injective.

**2.(b).** *Show that E is injective, but not surjective.* 

**3.** Suppose we have a two state Markov chain with stochastic matrix

3 10 points

$$P = \left(\begin{array}{cc} 0.1 & 0.5\\ 0.9 & 0.5 \end{array}\right)$$

Given the probability vector  $v = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$ , find  $\lim_{n \to \infty} P^n v$ .

**4.** Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$4$$
10 points

**4.(a).** Find the characteristic polynomial  $p_A(t)$  of A.

**4.(b).** *Find the eigenvalues of A.* 

**4.(c).** Find an orthonormal basis for each eigenspace of A in  $\mathbb{R}^3$ .

**4.(d).** Is A diagonalizable? If so, find a matrix  $S \in M_{3\times 3}(\mathbb{R})$  so that  $S^{-1}AS$  is diagonal. If not, explain.

**4.(e).** Is A diagonalizable with orthogonal matrices? If so, find an orthogonal matrix  $U \in M_{3\times 3}(\mathbb{R})$  so that  $U^T A U$  is diagonal. If not, explain.

**5.** Consider the following matrix:

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$



**5.(a).** What is the sum of the roots of the characteristic polynomial of B?

**5.(b).** What is the product of the roots of the characteristic polynomial of *B*?

**5.(c).** *Are the roots of the characteristic polynomial of B real?* 

6. Consider the two dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \left( \begin{array}{cc} 1.7 & 0.3 \\ 1.2 & 0.8 \end{array} \right)$$

**6.(a).** *Is the origin an attractor, repeller, or saddle point?* 

**6.(b).** *Find the directions of greatest attraction or repulsion.* 



7. Let 
$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$
,  $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ , and  $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ .

Find an orthonormal basis for the vector subspace of  $\mathbb{R}^4$  spanned by  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{x}_3$ , and  $\mathbf{x}_4$ .

**8.** Find the equation  $y = \beta_0 + \beta_1 x$  of the line that best fits the given data points, as a least squares model:

8 10 points

$$\left[\begin{array}{c} x\\ y\end{array}\right]: \quad \left[\begin{array}{c} -1\\ 0\end{array}\right], \left[\begin{array}{c} 0\\ 1\end{array}\right], \left[\begin{array}{c} 1\\ 2\end{array}\right], \left[\begin{array}{c} 2\\ 1\end{array}\right]$$

9. TRUE or FALSE. You do not need to justify your answer.

**9.(a).** Suppose *A* and *B* are invertible  $n \times n$  matrices, and that AB = BA. Then  $A^{-1}B^{-1} = B^{-1}A^{-1}$ .

9

T F

**9.(b).** Let  $f : V \to V$  be a linear map of a vector space to itself. If f is surjective, then f is an isomorphism.

T F

**9.(c).** Suppose that *P* is an  $n \times n$  matrix with positive entries, such that the column sums are equal to 1. Then  $\lim_{n\to\infty} P^n$  exists.

**9.(d).** Suppose that  $T : V \to V'$  is a linear map of finite dimensional vector spaces. Then  $\dim V' = \dim \ker(T) + \dim \operatorname{Im}(T)$ . T F |

**9.(e).** If an  $n \times n$  matrix has *n* distinct eigenvalues, then it has *n* linearly independent eigenvectors.

T F

**9.(f).** If *v* is an eigenvector for an  $n \times n$  matrix *A* with eigenvalue  $\lambda$ , and  $r \neq 0$  is a real number, then rv is an eigenvector for *A* with eigenvalue  $\lambda$ .

T F

**9.(g).** Suppose that  $A \in M_{n \times n}(\mathbb{R})$  is symmetric, and let  $v_1, v_2 \in \mathbb{R}^n$  be eigenvectors with corresponding eigenvalues  $\lambda_1, \lambda_2$ . If  $\lambda_1 \neq \lambda_2$ , then  $v_1$  is orthogonal to  $v_2$ .

T F

**9.(h).** Suppose that *M* is an  $n \times n$  matrix and  $M^N = 0$  for some integer N > 1. Then *M* is diagonalizable.

T F

**9.(i).** For an  $n \times n$  matrix A, if det(cof A) = 0, then det A = 0. T F

**9.(j).** Let  $v, w \in \mathbb{R}^n$ . If  $\theta$  is the angle between v and w, then  $\cos \theta = \frac{v.w}{||v||||w||}$ .

T F