MIDTERM
LINEAR ALGEBRA

MATH 2130
SUMMER 2018

Friday June 15, 2018
9:15 AM – 10:50 AM

Name ______________________________

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	
10	10	10	10	10	10	10	10	total

Date: June 14, 2018.

1. *Find all solutions to the following system of linear equations:*

1
10 points

$$\begin{array}{rcrcrcr} 3x_1 & + & 9x_2 & + & 27x_3 & = & -3 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 5 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & -4 \end{array}$$

SOLUTION

The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

To find this, we row reduce the associated augmented matrix

$$\left[\begin{array}{rrr|r} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right]$$

$$\begin{array}{r} R_1' = \frac{1}{3}R_1 \\ \\ R_3' = \frac{1}{2}R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 1 & 4 & 13 & -2 \end{array}\right]$$

$$\begin{array}{r} \\ R_2' = 3R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ 0 & -2 & -8 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right]$$

$$\begin{array}{rl} R_2' = R_3 & \mapsto \\ R_3' = R_2' & \qquad R_3'' = 2R_2 + R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1' = R_1 - 3R_2 \left[\begin{array}{rrr|r} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now we adjust the RREF:

$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Thus the solutions to the system of equations are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

as claimed.

To check your answer, you can confirm that these are in fact solutions; e.g., $(x_1, x_2, x_3) = (2, -1, 0)$ is a solution to the system of equations, and $(x_1, x_2, x_3) = (-3, 4, -1)$ is a solution to the following homogeneous system of equations:

$$\begin{array}{rcrcrcl} 3x_1 & + & 9x_2 & + & 27x_3 & = & 0 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 0 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & 0 \end{array}$$

2. Consider the matrix

2
10 points

$$A = \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{bmatrix}$$

2.(a). *Find the reduced row echelon form of A.*

2.(b). *Are the columns of A linearly independent?*

2.(c). *Are the rows of A linearly independent?*

2.(d). *What is the column rank of A?*

2.(e). *What is the row rank of A?*

SOLUTION

(a) The RREF of the matrix A is

$$\boxed{\text{RREF}(A) = \begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Indeed we have

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{bmatrix}$$

$$\begin{matrix} R_3' = -3R_1 + R_3 \\ R_4' = -R_1 + R_4 \end{matrix} \quad \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ R_3' = -\frac{1}{10}R_3 \\ R_4' = -R_2 + R_4 \end{array} \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r} R_1' = R_1 - 4R_3 \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The columns of A are **not linearly independent** since at most 4 vectors in $\mathbb{R}^4$ can be linearly independent.

(c) The rows of A are **not linearly independent**, since $\text{RREF}(A)$ has a zero row.

(d) The column rank is equal to the row rank, which is $\boxed{3}$, the number of nonzero rows in the RREF of A.

(e) The row rank is $\boxed{3}$.

3. Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L(x_1, x_2, x_3) = (2x_1 - x_3, 3x_2 + x_3).$$

Write down the matrix form of the linear map L.

3
10 points

SOLUTION

The matrix form of L is

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

We find this by computing L on the standard basis elements:

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 1 - 0 \\ 3 \cdot 0 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 0 \\ 3 \cdot (1) + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 1 \\ 3 \cdot 0 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These give the corresponding columns of the matrix form of L.

To check your answer, you can confirm that

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ 3x_2 + x_3 \end{bmatrix}$$

4. Consider the matrix

4
10 points

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

4.(a). *Find the inverse of B.*

4.(b). *Does there exist* $x \in \mathbb{R}^3$ *such that* $Bx = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$?

SOLUTION

(a) The solution is

$$B^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -3 & -1 & 6 \end{bmatrix}$$

To do this, we consider the augmented matrix $\left[\begin{array}{c|c} B & I \end{array}\right]$, and do row reduction until we arrive at the matrix $\left[\begin{array}{c|c} I & B^{-1} \end{array}\right]$. In more detail:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -6 & -1 & -3 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -6 & -1 & -3 & 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & -6 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & 6 \end{array}\right]$$

The matrix on the right is the matrix B^{-1}.

You can check your answer by computing:

$$BB^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -3 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) **YES** Since B is invertible, given any $b \in \mathbb{R}^3$, we have that $B(B^{-1}b) = b$. In particular, for $b = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$, we have that $x = B^{-1}\begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$ satisfies $Bx = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$.

5. Suppose that

5
10 points

$$C = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Find a lower triangular matrix L and an echelon form matrix U such that $C = LU$.

SOLUTION

An answer is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

We can find U, L, and L^{-1} (for the sake of exposition) directly in the following way. We consider:

$$\left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ -4 & -5 & 3 & -8 & 1 & 0 & 1 & 0 & 0 \\ 2 & -5 & -4 & 1 & 8 & 0 & 0 & 1 & 0 \\ -6 & 0 & 7 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & -9 & -3 & -4 & 10 & -1 & 0 & 1 & 0 \\ 0 & 12 & 4 & 12 & -5 & 3 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 7 & -5 & -4 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & -15 & -10 & -2 & 1 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

In other words, we perform row operations (only adding scalar multiples of higher rows to lower rows) on the augmented matrix $\left[\begin{array}{c|c} C & I \end{array}\right]$ putting C in echelon form. This process ends in $\left[\begin{array}{c|c} U & L^{-1} \end{array}\right]$.

We note, however, that it is elementary to compute L in this process. Each of the colored columns in L on the right is just the corresponding colored column in the matrix on the left, divided by the top colored entry in that column.

We can check our answer by computing:

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} = C$$

6. **TRUE** or **FALSE**: *Suppose that $V \subseteq \mathbb{R}^n$ is a subset satisfying:*

(1) *For all $v_1, v_2 \in V$, we have $v_1 + v_2 \in V$.*
(2) *For all $v \in V$, we have $-v \in V$.*

Then V is a subspace of $\mathbb{R}^n$.

6
10 points

SOLUTION

FALSE For instance, consider $\mathbb{Z}^n \subseteq \mathbb{R}^n$; i.e., the elements of $\mathbb{R}^n$ with integral coordinates. The subset $\mathbb{Z}^n$ satisfies both (1) and (2) above, but is not a subspace of $\mathbb{R}^n$, since, for example, $(1, \ldots, 1) \in \mathbb{Z}^n$, but $\frac{1}{2}(1, \ldots, 1) \notin \mathbb{Z}^n$.

7. The equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ (the Leontief Production Equation) arises in the Leontief Input-Output Model. Here $\mathbf{x}, \mathbf{d} \in M_{n\times 1}(\mathbb{R})$ are column vectors and $C \in M_{n\times n}(\mathbb{R})$ is a square matrix. Consider also the equation $\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$ (called the price equation), where $\mathbf{p}, \mathbf{v} \in M_{n\times 1}(\mathbb{R})$ are column vectors.

7
10 points

Show that

$$\mathbf{p}^T\mathbf{d} = \mathbf{v}^T\mathbf{x}.$$

(This quantity is known as GDP.) [Hint: Compute $\mathbf{p}^T\mathbf{x}$ in two ways.]

SOLUTION

We are given the equations

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$
$$\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$$

From the first equation we have

$$\begin{aligned}\mathbf{p}^T\mathbf{x} &= \mathbf{p}^T(C\mathbf{x} + \mathbf{d}) \\ &= \mathbf{p}^TC\mathbf{x} + \mathbf{p}^T\mathbf{d}\end{aligned}$$

Now, the transpose of the second equation, $\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$, is $\mathbf{p}^T = \mathbf{p}^TC + \mathbf{v}^T$, giving

$$\begin{aligned}\mathbf{p}^T\mathbf{x} &= (\mathbf{p}^TC + \mathbf{v}^T)\mathbf{x} \\ &= \mathbf{p}^TC\mathbf{x} + \mathbf{v}^T\mathbf{x}\end{aligned}$$

Putting the expressions for $\mathbf{p}^T\mathbf{x}$ together, we have

$$\mathbf{p}^TC\mathbf{x} + \mathbf{p}^T\mathbf{d} = \mathbf{p}^TC\mathbf{x} + \mathbf{v}^T\mathbf{x}$$

Subtracting $\mathbf{p}^TC\mathbf{x}$ from both sides, we arrive at

$$\mathbf{p}^T\mathbf{d} = \mathbf{v}^T\mathbf{x}$$

completing the proof.

8
10 points

8. **TRUE** or **FALSE**. You do **not** need to justify your answer.

8.(a). Let $A \in M_{m\times n}(\mathbb{R})$. There is an $x \in \mathbb{R}^n$ such that $Ax = 0$.

T F | TRUE: Take $x = 0$.

8.(b). Let $A \in M_{m\times n}(\mathbb{R})$. If the columns of A span $\mathbb{R}^m$, then for any $b \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^n$ such that $Ax = b$.

T F | TRUE: The image of the linear map is the column span.

8.(c). The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$ is a linear map.

T F | FALSE: $f(1) + f(1) \neq f(2)$.

8.(d). If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{bmatrix}$ for each natural number n.

T F | FALSE: $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

8.(e). If A and B are $m \times n$ matrices, then $A + B = B + A$.

T F | TRUE: We have $(A + B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B + A)_{ij}$.

8.(f). Let $A \in M_{m\times n}(\mathbb{R})$. If the rows of A are linearly independent, then for any $b \in \mathbb{R}^m$ there is at most one $x \in \mathbb{R}^n$ such that $Ax = b$.

T F | FALSE: Take $A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

8.(g). Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The kernel of f is a sub-vector space of $\mathbb{R}^n$.

T F | TRUE: We have seen this in class.

8.(h). If the columns of a square matrix A are linearly independent, then A^T is invertible.

T F | TRUE: This follows from our characterization of invertible matrices: A^T invertible $\iff$ A invertible $\iff$ columns of A are linearly independent.

8.(i). If A is a square matrix such that sums of the absolute values of the entries of each column of A is less than 1, then $(\mathrm{Id} - A)$ is invertible.

T F | TRUE: We saw this in class.

8.(j). Suppose that A and B are square matrices, and AB is invertible. Then A and B are invertible.

T F | TRUE: If AB is invertible, then as a linear maps, B is injective and A is surjective, which we have seen, for square matrices, is enough to show that the matrices are invertible.