§6.6 Part I: Work

(Created by Faan Tone Liu)

Solutions: 2/19/18

Key Points:

- $\boxed{W = \text{Force} \times \text{Distance} = F \cdot d}$
- Units:

	F = Force	d = Distance	W = Work
Metric	Newtons $N = \frac{kg\, m}{sec^2}$	Meters m	Newton-meters OR Joules $N \cdot m \stackrel{OR}{=} J$
U.S. Units	Pounds lbs	Feet ft	Foot pounds ft·lbs

- Now, what if F is not constant?

$$W = \int F\,dx \quad \text{OR} \quad W = \int \underbrace{dw}_{\text{work for a slice}}$$

- Dealing with springs - **Hooke's Law:**

$$F = kx\ ,$$

where x is the distance stretched or compressed past the natural (equilibrium) length, and k is the spring constant.

- Dealing with the force of gravity (metric system):

$$F_g = mg\ ,$$

where m is the mass of the object and $g = 9.8\frac{\text{m}}{\text{sec}^2}$.

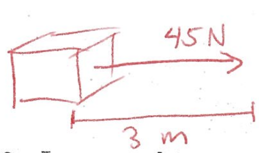
- Dealing with the force of graity (U.S. system):

$$F_g = \text{weight in lbs}\ .$$

Examples:

1. A box is slid 3 meters across a carpet against a force of kinetic friction of 45N. How much work is done?

Whole object moves same distance and acted on by same force. No calculus necessary!

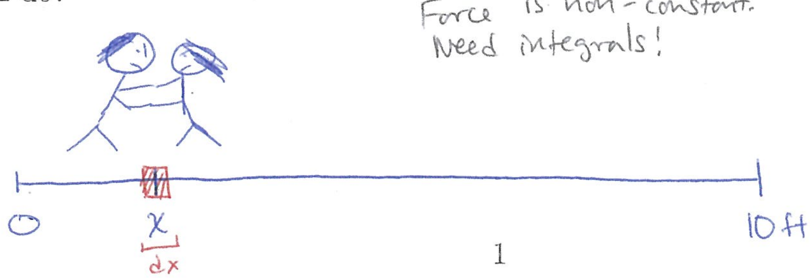


$$W = F \cdot d = 45N \cdot 3m = 135\ J$$

(Technically should be negative since box moving in opposite direction of force)

2. I am pushing my sister across a 10 foot room. She pushes back with increasing ferocity, with a force of $20 + \frac{x^2}{2}$ pounds, where x is how far I have pushed her. How much work do I do?

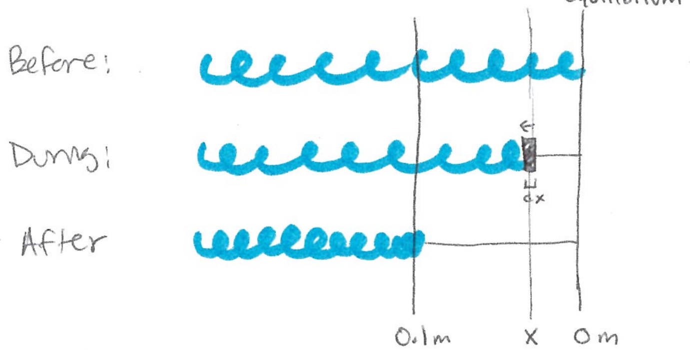
Force is non-constant. Need integrals!



$$W_{slice} = F \cdot d = \left(20 + \frac{x^2}{2}\right) \cdot dx$$

$$W = \int_0^{10} \left(20 + \frac{x^2}{2}\right) dx = \cdots = \boxed{366.\overline{6}\ \text{ft·lbs}}$$

3. A 30-centimeter long spring with a spring constant of $k = 120\frac{\text{N}}{\text{m}}$ is compressed to 20cm. Calculate the work done.

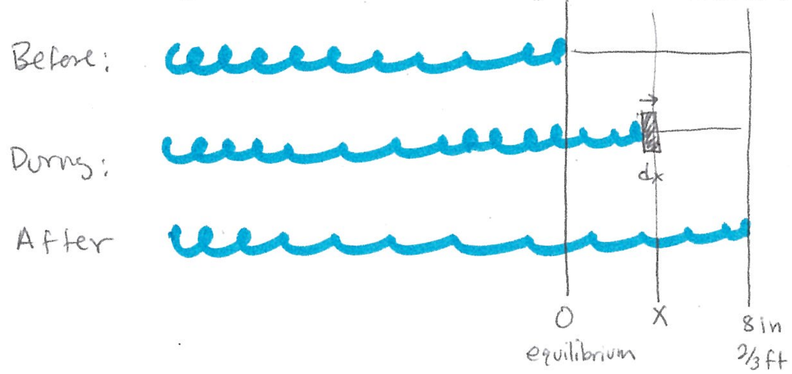


$W_{slice} = F\cdot d = kx\cdot dx = 120x\,dx$

distance from equilibrium

$W = \int_0^{.1} 120x\,dx = 60x^2\Big|_0^{.1} = 0.6\text{J}$

4. A force of 10 lbs is required to hold a spring stretched to 6 inches past its natural length. Calculate the work required to stretch it 8 inches past its natural length.



$W_{slice} = F\cdot d = kx\cdot dx = kx\,dx$

$W = \int_{0\text{ft}}^{2/3\text{ft}} kx\,dx = \frac{k}{2}x^2\Big|_0^{2/3}$ ⚠ Need k!

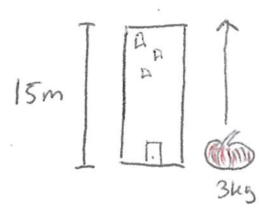
We know when $x = \frac{1}{2}$ft $F = 10$ lbs;

$10 = k\cdot\frac{1}{2} \Rightarrow k = 20$

$W = \frac{20}{2}x^2\Big|_0^{2/3}$

$= \boxed{44.4 \text{ ft}\cdot\text{lbs}}$

5. How much energy is required to hoist a 3-kilogram pumpkin 15 meters to the roof of the math building?



Force is constant for whole object so don't need to integrate (also distance the whole object travels is the same)

$F = F_g = mg$
$= 3\text{kg}\cdot 9.8\,\text{m}/\text{sec}^2$

$W = F_g\cdot d = mg\cdot 15\text{m} = 3\text{kg}\cdot 9.8\,\text{m}/\text{s}^2\cdot 15\text{m}$
$= \boxed{441\text{ Nm}}$

6. How much energy is required to carry a 44-lb stack of books up to the third floor of the math building? (30 ft.)

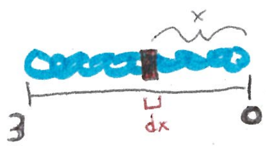
Again, Force + distance are the same for the whole object, so no integral required:

$W = F_g\cdot d = \underbrace{44\text{lbs}}_{\text{U.S. system, not metric}}\cdot 30\text{ft} = \boxed{1320\text{ ft}\cdot\text{lbs}}$

Could also do this by lifting (with integral) chain so that lower end at ground level, then add on $(6kg \cdot 9.8) \cdot 2m$ for last 2m [like pumping problem]

7. A 6-kg chain is 3 meters long. How much work is done lifting it from the ground until its lower end is 2 meters off of the ground?

Before:



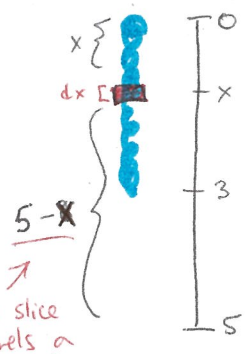
Some of the chain moves farther than other parts, so need an integral:

$$W_{slice} = F_g \cdot d = m \cdot g \cdot (5-x) = \rho\, dx \cdot 9.8\,(5-x)$$

mass of slice is $\rho \cdot dx$, where ρ is mass-density of chain

$$\rho = \frac{6\text{ kg}}{3\text{ m}} = 2\frac{\text{kg}}{\text{m}}$$

After



each slice travels a distance of 5-x

Slices "live" between x=0 and x=3

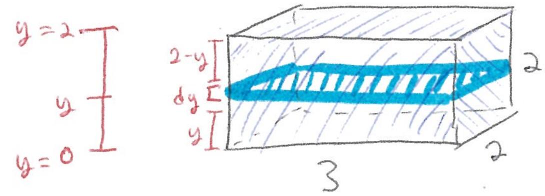
$$W = \int_0^3 2 \cdot dx \cdot 9.8\,(5-x) = \int_0^3 2 \cdot 9.8\,(5-x)\,dx$$

$$= \cdots = \boxed{205.8\text{ J}}$$

* For this problem, I picked a spot for x=0. There are many ways to do this (e.g. at other end of chain). You'll get same answer.

8. How much work is done emptying a $2 \times 2 \times 3$-ft rectangular tank? The water must be pumped to a point in the upper corner of the tank.

Some of the water travels farther than other parts, so need an integral.



$$W_{slice} = F_g \cdot d = V_{slice} \cdot \rho \cdot d = 6\,dy \cdot 62.5 \cdot (2-y)$$

Volume of slice $= 3 \times 2 \times dy$

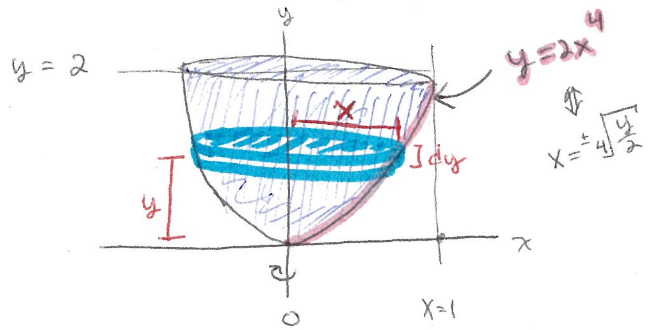
weight of H_2O $= 62.5 \frac{\text{lbs}}{\text{ft}^3}$

a slice moves 2-y feet.

* Note: could also put y=0 in other places.

$$W = \int_0^2 6 \cdot 62.5\,(2-y)\,dy = \cdots = \boxed{750\text{ ft} \cdot \text{lbs}}$$

9. A tub has the shape of the solid of revolution formed by rotating around the y-axis the portion of the curve $y = 2x^4$ that lies between $x = 0$ and $x = 1$. (Draw a picture.) How much work is done to empty the tank? All of the water must be pumped out of the top of the tank.



Each slice moves a different amount and each slice has a different amount of force acting on it (different volumes), so we need to do an integral.

$$W_{slice} = F_g \cdot d$$
$$= m \cdot g \cdot d$$
$$= V_{slice} \cdot \rho \cdot g \cdot (2-y)$$
$$= \pi \sqrt{\tfrac{y}{2}}\,dy \cdot 10^3 \tfrac{kg}{m^3} \cdot 9.8\, m/sec^2\, (2-y)$$
$$= 9800\pi \sqrt{\tfrac{y}{2}}\,(2-y)\,dy$$

ρ = mass density of water
$= 10^3 \frac{kg}{m^3}$

V_{slice} = Volume of slice
$= \pi x^2 dy$
$= \pi \left(\sqrt[4]{y/2}\right)^2 dy$
$= \pi \sqrt{\frac{y}{2}}\, dy$

$$W = \int_0^2 9800\pi \sqrt{\tfrac{y}{2}}\,(2-y)\,dy$$

Slices "live" between $y=0$ and $y=2$

$$= \cdots = \frac{31360}{3}\pi \approx 32840\,J$$