§5.7: Partial Fractions Decomposition

(Thanks to Faan Tone Liu)

Solutions: 1/30

Key Points:

- Use this method to integrate rational functions.
- * Degree of numerator must be lower than degree of denominator. If needed, start with long division.
- Factor the denominator
- Solve the decomposition according to the right form:
  - Linear Factors: $\dfrac{5x+3}{(x+1)(x+4)} = \dfrac{A}{x+1} + \dfrac{B}{x+4}$
  - Repeated Linear Factors: $\dfrac{2x-4}{(x-2)(x+3)^2} = \dfrac{A}{x-2} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$
  - Irreducible Quadratic Factors: $\dfrac{2x^2-3x-1}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$
  - Mixtures are possible!
- Make sure you remember how to integrate the "outputs" of the partial fractions decomposition. For example:

$$\frac{5}{2x+1},\ \frac{3}{(x+4)^2},\ \frac{2x}{x^2+9},\ \frac{4}{x^2+25},\ \text{or}\ \frac{3x+1}{x^2+16}.$$

- Other notes and tips:

Examples:

1. $\displaystyle\int \frac{2x^2+3x-3}{x^2-x}\,dx$

First, use long division to re-write the integrand so the fraction involved has a numerator with a lower degree than the denominator.

$$\begin{array}{r} 2 \\ x^2-x \,\big)\, \overline{2x^2+3x-3} \\ -(2x^2-2x) \\ \hline 5x-3 \end{array}$$

so $\dfrac{2x^2+3x-3}{x^2-x} = 2 + \dfrac{5x-3}{x^2-x}$

The form of the decomposition of $\frac{5x-3}{x(x-1)}$ is:

$$\frac{5x-3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Here are two methods to find A and B in the partial fractions decomposition:

Method I: Equate Coefficients	Method II: Judicious Substitution
$5x-3 = A(x-1) + Bx$	$5x-3 = A(x-1) + Bx$
$5x-3 = Ax - A + Bx$	$\underline{x=1}$: $5\cdot 1 - 3 = B\cdot 1$
$5x-3 = (A+B)x - A$	$2 = B$
$\begin{cases} 5 = A+B \\ -3 = -A \end{cases} \Rightarrow \boxed{\begin{array}{l} A=3 \\ B=2 \end{array}}$	$\underline{x=0}$: $5\cdot 0 - 3 = A(0-1) + B\cdot 0$
	$-3 = -A$
	$A = 3$

Now, $\displaystyle\int \frac{2x^2+3x-3}{x^2-x}\,dx = \int 2 + \frac{3}{x} + \frac{2}{x-1}\,dx$

$$= 2x + 3\ln|x| + 2\ln|x-1| + C$$

2. $\displaystyle\int \frac{x^2+x-5}{(x-2)(x-1)^2}\,dx$

degree on top : 2
degree on bottom : 3

2 < 3 ✓

$$\frac{x^2+x-5}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2+x-5 = A(x-1)^2 + B(x-1)(x-2) + C(x-2)$$

$\underline{x=1:}$ $1^2+1-5 = 0+0+C(-1)$

$-3 = -C$

$\boxed{C=3}$

$\underline{x=2:}$ $2^2+2-5 = A(2-1)^2+0+0$

$\boxed{1=A}$

$\underline{x=0:}$ $-5 = A+2B-2C$

$-5 = (1)+2B-2(3)$ [A=1, c=3]

$0 = 2B$

$\boxed{B=0}$

Some other pts ok here, too (x=1, x=2 are great choices. There is no clear third choice, so pick one.)

$$\int \frac{x^2+x-5}{(x-2)(x-1)^2}\,dx = \int \frac{1}{x-2} + \frac{3}{(x-1)^2}\,dx$$

$$= \boxed{\ln|x-2| - \frac{3}{x-1} + C}$$

3. $\displaystyle\int \frac{10}{(x+1)\underbrace{(x^2+9)}_{\text{irreducible quad.}}}\,dx$

degree on top: 0

degree on bottom: 3

$0 < 3$ ✓

$$\frac{10}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$10 = A(x^2+9) + (Bx+C)(x+1)$$

$\underline{x=-1}$: $10 = A\cdot 10 + 0 \Rightarrow \boxed{A=1}$

Now, $10 = x^2+9 + Bx^2 + Bx + Cx + C$

$$10 = (1+B)x^2 + (B+C)x + (9+C)$$

$$\begin{cases} 1+B=0 \\ B+C=0 \\ 9+C=10 \end{cases} \Rightarrow \boxed{\begin{array}{c} B=-1 \\ C=1 \end{array}}$$

$$\int \frac{10}{(x+1)(x^2+9)}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{-x+1}{x^2+9}\,dx$$

$$= \ln|x+1| + \int \frac{-x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x+1| + -\frac{1}{2}\int \frac{1}{u}\,du + \frac{1}{3}\arctan(x/3)$$

$$= \boxed{\ln|x+1| - \frac{1}{2}\ln|x^2+9| + \frac{1}{3}\arctan(x/3) + C}$$

$\underline{u\text{-sub}}$

$u = x^2+9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$