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# Math 2300-007: Integration By Parts

**Key points:**

- Integration by parts comes from the product rule for derivatives:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x) + g(x)f'(x)$$

These two statements are equivalent

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

subtract $\int g(x)f'(x)dx$ from both sides

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

let $u=f(x)$, $v=g(x)$, $du=f'(x)dx$, $dv=g'(x)dx$

$$\int u\,dv = u\cdot v - \int v\,du$$

- The integration by parts formula is given by

$$\boxed{\int u\,dv = uv - \int v\,du}$$

- Advice for choosing $u$ and $dv$:
  - pick $u$ so that $u'$ is simpler
  - pick $dv$ so that $v$ (i.e. antiderivative of $dv$) is simpler
  - If there's only one function e.g. $\int \arctan x\,dx$, let $u$ be that function and $dv=dx$

- Use integration by parts when...
  - The integral is a product
  - u/du-sub doesn't work

- Tips: - Use "boomerang" when you have two functions like $e^x$, $\sin x$, $\cos x$ that "repeat" when you integrate/differentiate (see #9)
  - Sometimes, need to use by parts twice (see #4) or in combo with u sub (#5)

Compute the following integrals using integration by parts:

1. $\int \underbrace{x}_{u} \underbrace{\cos x\,dx}_{dv} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$\left\{\begin{matrix} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x\end{matrix}\right\}$

2. $\int_0^2 xe^x\,dx = xe^x\Big|_0^2 - \int_0^2 e^x dx = [2e^2-0] - e^x\Big|_0^2 = 2e^2 - [e^2-e^0]$

$\left\{\begin{matrix} u=x & dv=e^x dx \\ du=dx & v=e^x\end{matrix}\right\}$

$= 2e^2 - e^2 + 1$

$= e^2+1$

3. $\displaystyle\int_4^9 \frac{\ln(y)}{\sqrt{y}}\,dy = \ln y \cdot 2\sqrt{y}\Big|_4^9 - \int_4^9 2y^{1/2}\cdot\frac{1}{y}\,dy = \ln(9)\cdot 2\sqrt{9} - \ln(4)\cdot 2\sqrt{4} - \int_4^9 2y^{-1/2}\,dy$

$= 6\ln(9) - 4\ln(4) - \left[4y^{1/2}\right]_4^9$

$= 6\ln(9) - 4\ln(4) - [12 - 8]$

$= 6\ln(9) - 4\ln(4) - 4$

$\left\{\begin{matrix} u = \ln y & dv = \frac{1}{\sqrt{y}}\,dy \\ du = \frac{1}{y}\,dy & v = 2y^{1/2} \end{matrix}\right\}$

4. $\displaystyle\int \theta^2 \sin(3\theta)\,d\theta = -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{3}\int \theta\cos(3\theta)\,d\theta = -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{3}\left[\frac{1}{3}\theta\sin(3\theta) - \int \frac{1}{3}\sin(3\theta)\,d\theta\right]$

$= -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{9}\theta\sin(3\theta) - \frac{2}{9}\int\sin(3\theta)\,d\theta$

$= -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{9}\theta\sin(3\theta) + \frac{2}{27}\cos(3\theta) + C$

#1 $\left\{\begin{matrix} u = \theta^2 & dv = \sin(3\theta)\,d\theta \\ du = 2\theta\,d\theta & v = -\frac{1}{3}\cos(3\theta) \end{matrix}\right\}$

#2 $\left\{\begin{matrix} u = \theta & dv = \cos(3\theta)\,d\theta \\ du = d\theta & v = \frac{1}{3}\sin(3\theta) \end{matrix}\right\}$

Tip: Choose $u = \theta^2$, the polynomial, so the power goes down

5. $\displaystyle\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\int\frac{1}{u}\,du$

$= x\arctan x - \frac{1}{2}\ln|u| + C$

$= \boxed{x\arctan x - \frac{1}{2}\ln|1+x^2| + C}$

$\left\{\begin{matrix} u = \arctan x & dv = dx \\ du = \frac{1}{1+x^2}\,dx & v = x \end{matrix}\right\}$

only one function, let $u$ = that function $dv = dx$

Do a u-sub!

$\left\{\begin{matrix} u = 1+x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{matrix}\right\}$

6. $\int \ln x \, dx$ $= x\ln x - \int x \cdot \frac{1}{x} dx = x\ln x - \int dx = x\ln x - x + C$

$\left\{ \begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{array} \right\}$

7. $\int (\ln x)^2 \, dx$ $= x(\ln x)^2 - \int 2\ln x \cdot \frac{1}{x} \cdot x \, dx = x(\ln x)^2 - 2\int \ln x \, dx$ see #6

$= x(\ln x)^2 - 2[x\ln x - x] + C$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

$\left\{ \begin{array}{ll} u = (\ln x)^2 & dv = dx \\ du = 2\ln(x) \cdot \frac{1}{x} dx & v = x \end{array} \right\}$

8. $\int_0^1 \frac{x+1}{e^x} dx$ $= \int_0^1 xe^{-x} dx + \int_0^1 e^{-x} dx = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx + \int_0^1 e^{-x} dx$

$= \left[\frac{-1}{e} + 0\right] + 2\int_0^1 e^{-x} dx$

$= \frac{-1}{e} + -2e^{-x}\Big|_0^1$

$= \frac{-1}{e} - 2\left[\frac{1}{e} - 1\right]$

$= \frac{-3}{e} + 2$

$\left\{ \begin{array}{ll} u = x & dv = e^{-x} dx \\ du = dx & v = -e^{-x} \end{array} \right\}$

"Boomerang"

This doesn't look simpler, but what if we do By Parts again...?

9. $\displaystyle\int e^t \cos t\, dt = e^t \sin t - \int e^t \sin t\, dt = e^t \sin t - \left[e^t(-\cos t) + \int e^t \cos t\, dt\right]$

By Parts #1 $\left\{ \begin{array}{ll} u = e^t & dv = \cos t\, dt \\ du = e^t dt & v = \sin t \end{array} \right\}$ BP #2 $\left\{ \begin{array}{ll} u = e^t & dv = \sin t\, dt \\ du = e^t dt & v = -\cos t \end{array} \right\}$

We have...

$$\boxed{\int e^t \cos t\, dt} = e^t \sin t + e^t \cos t - \boxed{\int e^t \cos t\, dt}$$

The blue boxes are the same! Solve for the blue box.

$$2\int e^t \cos t\, dt = e^t(\sin t + \cos t)$$

$$\boxed{\int e^t \cos t\, dt = \frac{1}{2} e^t (\sin t + \cos t) + C}$$

10. $\displaystyle\int \sec^3 x\, dx$

$[\tan^2 x + 1 = \sec^2 x]$

$$= \int \sec x \cdot \sec^2 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

Boomerang

$\left\{ \begin{array}{ll} u = \sec x & dv = \sec^2 x\, dx \\ du = \sec x \tan x & v = \tan x \end{array} \right\}$

We have... $\boxed{\int \sec^3 x\, dx} = \sec x \tan x - \boxed{\int \sec^3 x\, dx} + \int \sec x\, dx$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x\, dx$$

$$\boxed{\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C}$$

[ Aside: $\displaystyle\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$

$\left\{ \begin{array}{l} u = \sec x + \tan x \\ du = \sec x \tan x + \sec^2 x\, dx \end{array} \right\}$ $\displaystyle = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx = \int \frac{1}{u}\, du = \ln|\sec x + \tan x| + C$ ]

11. $\displaystyle\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{1 + \frac{1}{x^2}} \cdot \frac{1}{x}\, dx$

$\left\{ \begin{array}{ll} u = \arctan\left(\frac{1}{x}\right) & dv = dx \\ du = \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2}\, dx & v = x \end{array} \right\}$

$$= x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{x + \frac{1}{x}}\, dx$$

$$= x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2 + 1}\, dx$$

u-sub $\left\{ \begin{array}{l} u = x^2 + 1 \\ du = 2x\, dx \end{array} \right\}$

$$= x \arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \frac{1}{2} \int_{u=2}^{u=4} \frac{1}{u}\, du$$

$$= \left[\sqrt{3} \arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan(1)\right] + \frac{1}{2} \left[\ln|u|\right]_2^4$$

$$= \sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\left[\ln(4) - \ln(2)\right]$$

$$= \sqrt{3} \cdot \pi/6 - \pi/4 + \frac{1}{2} \ln\left(\frac{4}{2}\right) = \boxed{\sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4} + \ln(\sqrt{2})}$$