Solutions: 4/13/18

§7.3: Separable Differential Equations

Key Points:

- A **separable differential equation** is a differential equation that can be written in the form

$$f(y) \cdot \frac{dy}{dx} = g(x).$$

- To solve a separable differential equation:
  1. Separate the variables
  2. Integrate both sides (Remember $+C$!!!)
  3. Solve for y (if possible)
  4. Use the initial condition to find C.

- Other notes:

Examples:

1. Solve the differential equation $\frac{dy}{dx} = -2y$ if $y(0) = 1$.

$\frac{dy}{dx} = -2y$

$\frac{1}{y}\,dy = -2\,dx$

$\int \frac{1}{y}\,dy = \int -2\,dx$

$\ln|y| = -2x + C$

Solve for C:

$\ln|1| = -2(0) + C$

$0 = 0 + C$

$\ln|y| = -2x + 0$

$|y| = e^{-2x}$

$y = \pm e^{-2x}$

$\boxed{y = +e^{-2x}}$ since our initial condition was (0,1).

2. Solve the differential equation $\frac{dx}{dt} + x = 1$ if $x(1) = 0.1$.

$\frac{dx}{dt} = 1 - x$

$\frac{1}{1-x}\,dx = dt$

$\int \frac{1}{1-x}\,dx = \int dt$

⚠ Chain Rule (or u-sub)

$-\ln|1-x| = t + C$

Solve for C:

$-\ln|1-0.1| = (1) + C$

$C = -\ln(0.9)$

$-\ln(1-x) = t - \ln(0.9)$

$\ln(1-x) = -t + \ln(0.9)$

$1 - x = e^{-t + \ln(0.9)}$

$\boxed{x = 1 - e^{-t + \ln(0.9)}}$

3. Solve the differential equation $\frac{du}{dt} = u + ut^2$ if $u(0) = 5$.

$$\frac{du}{dt} = u(1+t^2)$$
$$\frac{1}{u}\,du = (1+t^2)\,dt$$
$$\int \frac{1}{u}\,du = \int (1+t^2)\,dt$$
$$\ln|u| = t + \tfrac{1}{3}t^3 + C$$
$$|u| = e^{t+\frac{1}{3}t^3+C}$$

solve for C:

$$|5| = e^{(0)+\frac{1}{3}(0)^3+C}$$
$$5 = e^{0+C}$$
$$5 = e^C$$
$$C = \ln(5)$$

$$\rightarrow |u| = e^{t+\frac{1}{3}t^3+\ln(5)}$$
$$u = \pm e^{t+\frac{1}{3}t^3+\ln(5)}$$
$$\boxed{u = +e^{t+\frac{1}{3}t^3+\ln(5)}}$$

Use initial condition (0,5) to choose the ⊕ branch.

4. Solve the differential equation $\frac{dy}{dx} = xe^y$ if $y(0) = 0$.

$$\frac{dy}{dx} = xe^y$$
$$\frac{1}{e^y}\,dy = x\,dx$$
$$\int e^{-y}\,dy = \int x\,dx$$
$$-e^{-y} = \tfrac{1}{2}x^2 + C$$

Solve for C:

$$-e^{0} = \tfrac{1}{2}(0)^2 + C$$
$$-1 = 0 + C$$
$$C = -1$$

$$\rightarrow -e^{-y} = \tfrac{1}{2}x^2 - 1$$
$$e^{-y} = 1 - \tfrac{1}{2}x^2$$
$$-y = \ln\left(1 - \tfrac{1}{2}x^2\right)$$
$$\boxed{y = -\ln\left(1 - \tfrac{1}{2}x^2\right)}$$

5. Solve the differential equation $\frac{ds}{d\theta} = -s^2 \tan\theta$ if $s(0) = 2$.

$$\frac{ds}{d\theta} = -s^2 \tan\theta$$
$$-\frac{1}{s^2}\,ds = \tan\theta\,d\theta$$
$$\int -\frac{1}{s^2}\,ds = \int \tan\theta\,d\theta$$
$$\int -s^{-2}\,ds = \int \frac{\sin\theta}{\cos\theta}\,d\theta \quad \left\{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right\}$$
$$s^{-1} = \int -\frac{1}{u}\,du$$
$$s^{-1} = -\ln|u| + C$$
$$\frac{1}{s} = -\ln|\cos\theta| + C$$

Solve for C:

$$\frac{1}{(2)} = -\ln|\cos(0)| + C$$
$$\frac{1}{2} = -\ln|1| + C$$
$$\frac{1}{2} = 0 + C$$
$$C = \frac{1}{2}$$

$$\rightarrow \frac{1}{s} = -\ln|\cos\theta| + \frac{1}{2}$$
$$\boxed{s = \frac{1}{\frac{1}{2} - \ln|\cos\theta|}}$$

6. Find an equation of the curve that passes through the point (0, 1) and whose slope at (x, y) is xy.

Slope $= \frac{dy}{dx} = xy$

$\frac{1}{y}dy = x\,dx$

$\int \frac{1}{y}dy = \int x\,dx$

$\ln|y| = \frac{1}{2}x^2 + C$

Solve for C:

use (0,1):

$\ln|1| = \frac{1}{2}(0)^2 + C$

$0 = 0 + C$

$C = 0$

$\rightarrow \ln|y| = \frac{1}{2}x^2 + 0$

$|y| = e^{\frac{1}{2}x^2}$

$y = \pm e^{\frac{1}{2}x^2}$

$\boxed{y = +e^{\frac{1}{2}x^2}}$

Use ⊕ because of initial condition

7. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.

Notice that

$y' = u$,

so $\frac{dy}{dx} = u$.

Also, $y = u - x$,

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{du}{dx} - 1 = u$

$\frac{du}{dx} = u + 1$

$\frac{1}{u+1}du = dx$

$\int \frac{1}{u+1}du = \int dx$

$\ln|u+1| = x + C$

$|u+1| = e^{x+C}$

$u+1 = \pm e^{x+C}$

$u = -1 \pm e^{x+C}$

sub back in: $y = u - x$

$u = x + y$

$x + y = -1 \pm e^{x+C}$

$\boxed{y = -1 - x \pm e^{x+C}}$

8. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enteres the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

We want an equation $s(t)$ for the amount of salt at time t.

First, let's understand the <u>rates</u> we are given:

$$\underbrace{\frac{ds}{dt}}_{\text{Change in salt over time}} = \text{Change from brine entering} - \text{Change from brine leaving}$$

Change from brine leaving: current salt concentration $\left(\frac{kg}{L}\right) \cdot \frac{L}{min}$

$$\frac{ds}{dt} = \underbrace{\overset{kg/L \;\cdot\; L/min}{(0.03)\cdot 25}}_{kg/min} - \underbrace{\frac{s}{5000}\cdot 25}_{kg/min}$$

$$\frac{ds}{dt} = \left(0.03 - \frac{s}{5000}\right)25 \quad \leftarrow \text{units are } \frac{kg}{min}$$

Now, we'll use separation of variables to solve for s...

$$\frac{ds}{dt} = 25\left(0.03 - \frac{s}{5000}\right)$$

$$\frac{1}{5000}\cdot\frac{1}{0.03 - s/5000}\,ds = 25\,dt \cdot \frac{1}{5000}$$

$$\int \frac{1}{150 - s}\,ds = \int 0.005\,dt$$

$$-\ln|150 - s| = 0.005t + C$$

We know $s(0) = 20$ kg

<u>solve for C:</u>

$$-\ln|150 - 20| = 0.005(0) + C$$

$$-\ln|130| = C$$

$$C \approx -4.8675$$

$$-\ln|150 - s| = 0.005t - 4.8675$$

$$\ln|150 - s| = 4.8675 - 0.005t$$

$$|150 - s| = e^{4.8675 - 0.005t}$$

$$150 - s = \pm e^{4.8675 - 0.005t}$$

$$s = 150 \pm e^{4.8675 - 0.005t} \quad \leftarrow \text{use "}-\text{" because of initial cond.}$$

$$s = 150 - e^{4.8675 - 0.005t}$$

$\boxed{\text{So, after 30 mins, there are } s(30) \approx 38.1 \text{ kg of salt}}$

<u>Note:</u> brine entering is saltier than brine leaving, so this makes sense.