# §8.1: Sequences

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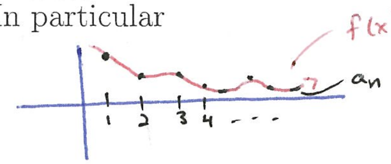
**Key Points:**

- Think of a sequence as a comma-separated list:

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

- A sequence is a function whose domain is the positive integers. You can graph a sequence of real numbers.
- We are often interested in the end behavior of a sequence, $\lim_{n\to\infty} a_n$. Hint: use the "connect the dots" function defined on $\mathbb{R}$ (i.e. $f(n) = a_n$). In particular




$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x),$$

and we can use Calc I tools like L'Hôpital's Rule.

- Some neat tools:
  - Squeeze Law (Sandwhich Theorem):

    If $a_n \le b_n \le c_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

  - Showing an alternating sequence converges:

    If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

  - Showing an alternating sequence diverges:

    If $\lim_{n\to\infty} a_n \neq 0$, then $(-1)^n a_n$ diverges.

    [Recall $(-1)^n a_n$, $a_n \ge 0$ is the form of an alternating series.]

  - Three ways to show a sequence is decreasing

    [Note, Showing $\lim_{n\to\infty} a_n \neq 0$ is Showing $\lim_{n\to\infty} |(-1)^n a_n| \neq 0$.]

    1. Show $\frac{a_{n+1}}{a_n} < 1$
    2. Show $a_{n+1} - a_n < 0$
    3. Use $f(x)$, where $f(n) = a_n$ and show $f'(x) < 0$

  - Note: Bounded Monotonic (i.e. increasing or decreasing) sequences must converge.

**Examples:**

1. Consider the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, a_n, \ldots$. Find a formula for $a_n$.

Denominators are $1, 3, 5, 7, \ldots, 2n-1, \ldots$

$a_n = \frac{1}{2n-1}$ [Note: $\lim_{n\to\infty} a_n = 0$, so seq. converges to 0.]

2. Consider the sequence $\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, \frac{1}{18}, \ldots, a_n, \ldots$. Find $a_n$.

3 5 7

2 2 ← constant second difference, so quadratic denominator

$a_n = \frac{1}{n^2+2}$ [Note: $\lim_{n\to\infty} \frac{1}{n^2+2} = 0$, so seq. conv. to 0.]

3. Consider the sequence $\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \ldots, a_n, \ldots$. Find $a_n$.

- Numerator: $2, 4, 6, \ldots, 2n, \ldots$
- Denom: $3, 3^2, 3^3, \ldots, 3^n, \ldots$

$a_n = \frac{2n}{3^n}$

[Note: Let $f(x) = \frac{2x}{3^x}$. Then $\lim_{x\to\infty} \frac{2x}{3^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{3^x \ln(3)} = 0$, (L'Hôpital)

So $\lim_{n\to\infty} a_n = 0$.

4. Consider the sequence $-\frac{5}{2}, \frac{8}{4}, -\frac{11}{8}, \frac{14}{16}, \ldots, a_n, \ldots$. Find $a_n$.

- Alternating, so $(-1)^n$
- Numerator: $5, 8, 11, 14, \ldots, 2+3n, \ldots$
- Denom: $2, 2^2, 2^3, \ldots, 2^n, \ldots$

$a_n = \frac{(-1)^n(2+3n)}{2^n}$

[Note: $|a_n| = \frac{2+3n}{2^n}$, let $f(x) = \frac{2+3x}{2^x}$.

Then, $\lim_{x\to\infty} f(x) \overset{H}{=} \lim_{x\to\infty} \frac{3}{2^x \ln 2} = 0$, (L'Hôp.)

So the seq. converges to 0.]

5. Consider the sequence $7, -\frac{9}{2}, \frac{11}{6}, -\frac{13}{24}, \ldots, a_n, \ldots$. Find $a_n$.

- Alternating: $(-1)^{n+1}$
- Numerator: $2n+5$
- Denominator: $1, 1\cdot 2, 1\cdot 2\cdot 3, 1\cdot 2\cdot 3\cdot 4, \ldots, n!$ ($n$ "factorial")

$a_n = \frac{(-1)^{n+1}(2n+5)}{n!}$

[Note: $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n+5}{n!} = \lim_{n\to\infty} \frac{2n}{n!} + \lim_{n\to\infty} \frac{5}{n!}$

$= \lim_{n\to\infty} \frac{2}{(n-1)!} + \lim_{n\to\infty} \frac{5}{n!} = 0 + 0 = 0$

So $a_n \to 0$

6. Suppose $a_n = \frac{\cos n}{n^2}$. Find $\lim_{n\to\infty} a_n$.

Can use squeeze Thm! $-1 \le \cos(n) \le 1$

$$\frac{-1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$$

Since $\lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0$, by the squeeze thm, $\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0$.

7. Suppose $a_n = \frac{(-1)^n \ln n}{n}$. Find $\lim_{n\to\infty} a_n$.

Consider $|a_n| = \left|\frac{\ln(n)}{n}\right| = \frac{\ln(n)}{n}$ for $n \ge 1$

Now, let $f(x) = \frac{\ln(x)}{x}$. Then, $\lim_{x\to\infty} f(x) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$, l'Hopital

so $\lim_{n\to\infty} a_n = 0$.

8. Suppose $a_n = \frac{(-1)^n (n^3+3)}{2n^3-1}$. Find $\lim_{n\to\infty} a_n$.

Consider $|a_n| = \frac{n^3+3}{2n^3-1}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3+3}{2n^3-1} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1+\frac{3}{n^3}}{2-\frac{1}{n^3}} = \frac{1}{2}.$$

⚠ $\lim_{n\to\infty} a_n$ does not exist! (Note: $\lim_{n\to\infty} |a_n|$ exists and is $\frac{1}{2}$)

9. Suppose $a_n = \frac{\sqrt{3n^2+4}}{n-1}$. Find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{3n^2+4}}{n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{3n^2+4}\sqrt{\frac{1}{n^2}}}{1-\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{3+\frac{4}{n^2}}}{1-\frac{1}{n}} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

10. Suppose $a_n = \left(1+\frac{1}{n}\right)^n$. Find $\lim_{n\to\infty} a_n$.

Consider $f(x) = \left(1+\frac{1}{x}\right)^x$.

$\lim_{x\to\infty} f(x)$ has form $1^\infty$, which is indeterminate! Time for l'Hôpital's Rule!

Let $L = \lim_{x\to\infty} f(x)$

$\ln(L) = \ln(\lim_{x\to\infty} f(x))$

$= \lim_{x\to\infty} \ln(f(x))$

$= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$ "form $\infty \cdot 0$"

$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$ "form $\frac{0}{0}$"

$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(\frac{-1}{x^2}\right)}{-\frac{1}{x^2}}$ ⚠ chain rule on top!

$= \lim_{x\to\infty} \frac{-1}{1+\frac{1}{x}} = -1$

Hence, $\ln(L) = -1$, so $L = e^{-1}$. We have $\boxed{\lim_{n\to\infty} a_n = \frac{1}{e}}$

11. Show $a_n = \dfrac{3^{n+2}}{5^n}$ is decreasing.

$$\frac{a_{n+1}}{a_n} = \frac{3^{(n+1)+2}}{5^{(n+1)}} \cdot \frac{5^n}{3^{n+2}} = \frac{3^{n+3} \cdot 5^n}{5^{n+1} \cdot 3^{n+2}} = \frac{3}{5} < 1,$$

so $a_n$ is decreasing. $[a_{n+1} < a_n]$

12. Show $a_n = \dfrac{n}{n+1}$ is increasing.

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} > 0 \text{ for all } n.$$

so $a_n$ is increasing. $[a_{n+1} > a_n]$

13. Show $a_n = \dfrac{n}{e^n}$ is decreasing.

Let $f(x) = \dfrac{x}{e^x}$

$f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{e^x(1-x)}{e^x \cdot e^x} = \dfrac{(1-x)}{e^x}$. For $x > 1$, this is negative, so $f(x)$ is decreasing on $(1, \infty)$. Hence $a_n$ is decreasing.

14. Find a formula for $a_n$ if $a_1 = 2$ and $a_{n+1} = a_n + 5$.

$a_1 = 2$

$a_2 = 2 + 5 = 7$

$a_3 = 7 + 5 = 12$

$a_4 = 12 + 5 = 17$

pattern is:

$a_n = 2 + 5(n-1) = 5n - 3$.

15. Find a formula for $a_n$ if $a_1 = 4$ and $a_{n+1} = 3 \cdot a_n$.

$a_1 = 4$

$a_2 = 3 \cdot 4$

$a_3 = 3 \cdot 3 \cdot 4 = 3^2 \cdot 4$

$a_4 = 3 \cdot 3^2 \cdot 4 = 3^3 \cdot 4$

$a_5 = 3 \cdot 3^3 \cdot 4 = 3^4 \cdot 4$

pattern is:

$a_n = 3^{n-1} \cdot 4$