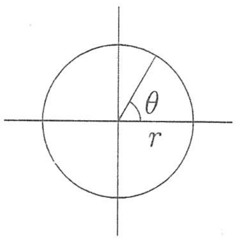
Calculus of Polar Curves (Appendix H2)

Thanks to Faan Tone Liu

Key Points:

- Area of a sector:

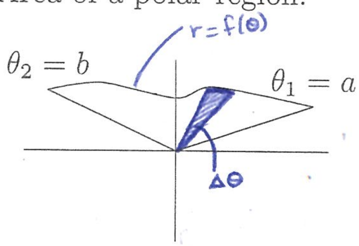


Area of entire circle = πr^2

Fraction of circle = $\frac{\theta}{2\pi}$

Area of sector = $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

- Area of a polar region:



Area of thin slice = $\frac{1}{2}[r(\theta)]^2 \Delta\theta$

Estimate of area = $\sum_{i=1}^{n} \frac{1}{2}[r(\theta_i)]^2 \Delta\theta$

Exact area = $\int_a^b \frac{1}{2}[r(\theta)]^2 d\theta$

- To find the slopes of tangent lines to polar curves and arc length of polar curves, use parametric equations:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{r(-\sin\theta) + \frac{dr}{d\theta}\cos\theta}$$

Arc length = $\int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

Arc length (simplified) = $\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

- Other Notes:

To get simplified arc length formula:

$$\int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{(-r\sin\theta + r'\cos\theta)^2 + (r\cos\theta + r'\sin\theta)^2}\, d\theta$$

$$= \int_a^b \sqrt{r^2\sin^2\theta - 2rr'\sin\theta\cos\theta + (r')^2\cos^2\theta + r^2\cos^2\theta + 2rr'\cos\theta\sin\theta + (r')^2\sin^2\theta}\, d\theta$$

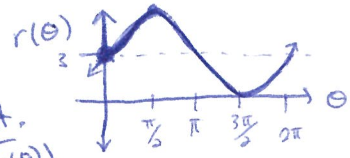
$$= \int_a^b \sqrt{r^2(\sin^2\theta + \cos^2\theta) + (r')^2(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= \int_a^b \sqrt{r^2 + (r')^2}\, d\theta.$$

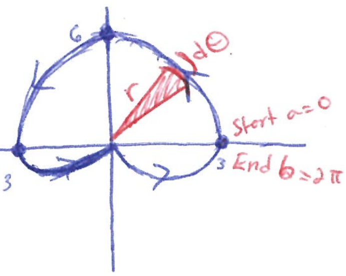
Examples:

1. Find the area inside the region bounded by $r = 3 + 3\sin\theta$

rect. (just $r(\theta)$ vs θ)



polar (graph of $r=3+3\sin\theta$)



$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(3+3\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 9 + 18\sin\theta + 9\sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 9 \, d\theta + 9\int_0^{2\pi} \sin\theta \, d\theta + \frac{9}{2}\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= 9\pi + 9(-\cos\theta)\Big|_0^{2\pi} + \frac{9}{2}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2} \, d\theta$$

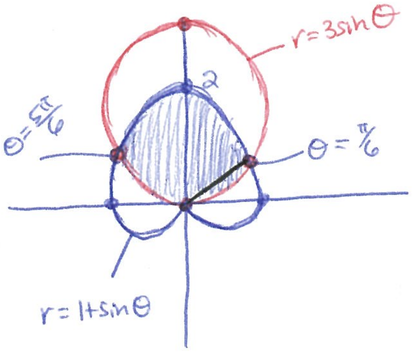
$$= 9\pi + 0 + \frac{9}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

$$= 9\pi + \frac{9}{2}\pi$$

$$= \frac{27}{2}\pi$$

2. Find the area of the region that lies inside both $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

$r = 1 + \sin\theta$ is a scaled version of the one above



$r = 3\sin\theta$ is a circle (plot points or draw a graph of $r(\theta)$ vs θ to see)

Need intersection points:

$$1 + \sin\theta = 3\sin\theta$$

$$1 = 2\sin\theta$$

$$\frac{1}{2} = \sin\theta$$

$$\theta = \pi/6, \ 5\pi/6$$

$$\text{Area} = \int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2 \, d\theta + \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2 \, d\theta + \int_{5\pi/6}^{\pi} \frac{1}{2}(3\sin\theta)^2 \, d\theta$$

[Think about which radius (i.e. blue or red) we are using.]

$A = \cdots = \frac{5\pi}{4}$ (Fun calculation that's good practice)

3. Find the length of $r = 2\csc\theta$ from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$. What is the slope of the curve at $x = \frac{\pi}{2}$?

$$x = r\cos\theta = 2\csc\theta \cdot \cos\theta = 2\cot\theta$$

$$y = r\sin\theta = 2\csc\theta \cdot \sin\theta = 2$$

$$\frac{dx}{d\theta} = -2\csc^2\theta$$

$$\frac{dy}{d\theta} = 0$$

Arc length $= \displaystyle\int_{\pi/6}^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\pi/6}^{\pi/2} \sqrt{4\csc^4\theta}\, d\theta$

Δ $\sin^2\theta$ is positive between $\pi/6$ and $\pi/2$

$$= \int_{\pi/6}^{\pi/2} 2\csc^2\theta\, d\theta$$

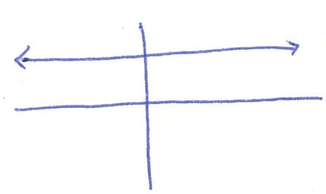
$$= -2\cot\theta \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{-2\cos(\pi/2)}{\sin(\pi/2)} + \frac{2\cos(\pi/6)}{\sin(\pi/6)} = 0 + \frac{2\cdot\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{2\sqrt{3}}$$

Slope: $\displaystyle\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{0}{-2\csc^2\theta} = 0$

Slope everywhere (and hence at $x = \pi/2$) is 0.

This function is a horizontal line!



$y=2$.

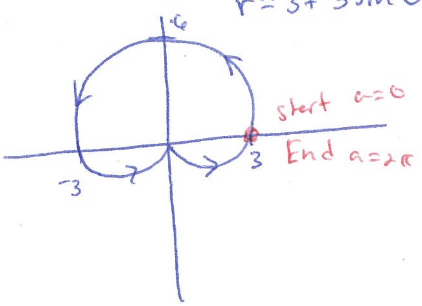
(Can figure this out from $\frac{dy}{dx} = 0$, from plotting points, or from converting to rect. coords:

$r = 2 \cdot \frac{1}{\sin\theta} \Rightarrow \underbrace{r\sin\theta}_{y} = 2$)

4. Find the arc length of the cardioid $3 + 3\sin\theta$.

See picture in #1 ~~if desired.~~ (You should).

$r = 3 + 3\sin\theta$



$$x = r\cos\theta = (3+3\sin\theta)\cos\theta = 3\cos\theta + 3\sin\theta\cos\theta$$
$$y = r\sin\theta = (3+3\sin\theta)\sin\theta = 3\sin\theta + 3\sin^2\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta + 3\sin\theta(-\cos\theta) + 3\cos^2\theta$$
$$\frac{dy}{d\theta} = 3\cos\theta + 6\sin\theta\cdot\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = 9\sin^2\theta + 9\sin^2\theta\cos\theta - 9\sin\theta\cos^2\theta$$
$$+ 9\sin^2\theta\cos\theta + 9\sin^2\theta\cos^2\theta - 9\sin\theta\cos^3\theta$$
$$+ -9\sin\theta\cos^2\theta - 9\sin\theta\cos^3\theta + 9\cos^4\theta$$

$$+ \quad \left(\frac{dy}{d\theta}\right)^2 = 9\cos^2\theta + 36\sin\theta\cos^2\theta + 36\sin^2\theta\cos^2\theta$$

Really Nasty... Try the other formula for arc length...

$$\text{Arc length} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(3+3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{9 + 18\sin\theta + 9\sin^2\theta + 9\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{9 + 18\sin\theta + 9}\, d\theta$$

Δ: $\sin^2\theta + \cos^2\theta = 1$

$$= \int_0^{2\pi} \sqrt{18 + 18\sin\theta}\, d\theta = \sqrt{18}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta = \sqrt{18}\int_0^{2\pi} \frac{\sqrt{1+\sin\theta}\,\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \sqrt{18}\int_0^{2\pi} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta = \sqrt{18}\int_0^{2\pi} \frac{\sqrt{\cos^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta = \sqrt{18}\int_0^{2\pi} \frac{|\cos\theta|}{\sqrt{1-\sin\theta}}\, d\theta$$

Now, do u-sub and break integral into regions where $\cos\theta$ is + or −, get ... $= \boxed{24}$