# Calculus of Parametric Eq. (§1.7, 3.4, 6.4)

Thanks to Faan Tone Liu

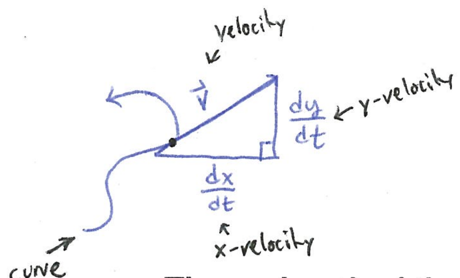
**Key Points:**

- Suppose $x(t)$ and $y(t)$ are parametric equations and $t$ represents time. Then
  - $\frac{dx}{dt}$ represents instantaneous velocity in x-direction (at time t).
  - $\frac{dy}{dt}$ represents instantaneous velocity in y-direction (at time t).
  - $\frac{dy}{dx}$ represents the slope of the tangent line to the curve (at time t).
- Formulae for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $t$ are

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)},$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{d}{dt}x(t)} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}.$$

- The instantaneous speed (of the snail) along the curve as a function of $t$ is:




Recall speed = |velocity|

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

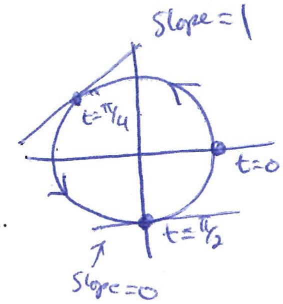
$$= \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

- The $\underbrace{\text{arc length}}_{\text{total dist. travelled}}$ of the curve for $t$ in $[a, b]$ is the integral of speed:

$$\text{Arc Length} = \int_a^b \text{speed} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- Other Notes:

**Examples:**




1. Consider the parametric curve given by $\begin{cases} x = 5\cos(3t) \\ y = 5sin(3t). \end{cases}$

   At $t = \frac{\pi}{4}$ and $t = \frac{\pi}{2}$, find the slope of the tangent line and the speed.

   $x'(t) = -5\sin(3t)\cdot 3 = -15\sin(3t)$
   
   $y'(t) = 5\cos(3t)\cdot 3 = 15\cos(3t)$

   Slope of tangent:

   $$\left.\frac{dy}{dx}\right|_{\pi/4} = \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{+15\cos(\frac{3\pi}{4})}{-15\sin\frac{3\pi}{4})} = \frac{15(-\frac{\sqrt{2}}{2})}{-15(\frac{\sqrt{2}}{2})} = 1$$

   $$\left.\frac{dy}{dx}\right|_{\pi/2} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{15\cos(\frac{3\pi}{2})}{-15\sin(\frac{3\pi}{2})} = \frac{0}{-15(-1)} = 0$$

   Speed:

   $\frac{\pi}{4}: \sqrt{[x'(\pi/4)]^2 + [y'(\pi/4)]^2} = \cdots = 15$

   $\pi/2: \sqrt{[x'(\pi/2)]^2 + [y'(\pi/2)]^2} = \cdots 15$

2. A half-line is parameterized by $\begin{cases} x = 2 + 3t \\ y = -1 + 5t, \end{cases}$ where $t \geq 0$.

   (a) Does (5,4) lie on the ray?

   $5 = x(t) = 2 + 3t \Rightarrow t = 1$
   
   $4 = y(t) = -1 + 5t \Rightarrow t = 1$

   Yes! At time $t=1$, the snail is at $(5,4)$.

   (b) Does (2,1) lie on the ray?

   $2 = x(t) = 2 + 3t \Rightarrow t = 0$
   
   $1 = y(t) = -1 + 5t \Rightarrow t = \frac{2}{5}$

   No! The only time when the x-coord is 2 is when $t=0$. At this time, the y-coord is $y(0) = -1$.

   (c) Does (-1,-6) lie on the ray?

   $-1 = x(t) = 2 + 3t \Rightarrow t = -1$

   No! Since $t < 0$, there is no time when the x-coord. is $-1$ and $t \geq 0$.

   (d) When does the line hit the $y$-axis?

   $0 = x = 2 + 3t \Rightarrow t = -2/3$. Since $-2/3 < 0$, the ray never hits the y-axis.

   (e) What is the speed of motion along the line?

   $\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 5$, speed $= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{9+25} = \sqrt{34}$

   (Since both eqns are linear, the speed is constant.)

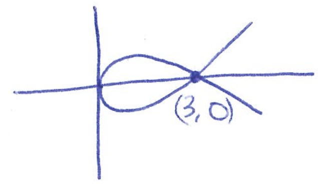
   (f). What is the slope of the line?

   $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3}{5}$

   (This doesn't depend on time either, since both components are linear and the parametric equation is a line.)

3. Use technology to graph

$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases} \quad ; \ t \in \mathbb{R}.$$




(a) Find equations for the tangent lines to the curve at (3,0).

Slope: $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-3}{2t}$ Need $t$-values...

pt: $(3,0)$

$3 = t^2 \Rightarrow t = \pm\sqrt{3}$

$0 = t^3 - 3t \Rightarrow 0 = t(t^2-3) \Rightarrow t = 0, \pm\sqrt{3}$

Line 1: $t = -\sqrt{3}$ $\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{9-3}{-2\sqrt{3}} = \frac{6}{-2\sqrt{3}} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$ $\boxed{y - 0 = -\sqrt{3}(x-3)}$

Line 2: $t = \sqrt{3}$ $\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{9-3}{2\sqrt{3}} = \cdots = \sqrt{3}$ $\boxed{y - 0 = \sqrt{3}(x-3)}$

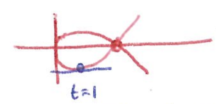
The two tangent lines.

(b) At $t = 1$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Check the concavity.

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3\cdot1^2-3}{2\cdot1} = \frac{0}{2} = 0$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{d}{dt}x(t)} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{2t(6t)-(3t^2-3)\cdot2}{(2t)^2}}{2t} = \frac{12t^2-6t^2+6}{(2t)^3} = \frac{6(t^2+1)}{(2t)^3}$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{6(1^2+1)}{(2\cdot1)^3} = \frac{6\cdot2}{8} = \frac{3}{2} > 0$, so concave up

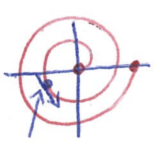



tangent line has slope 0 and is concave up.

4. Consider the parametric curve given by $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$, where $0 \le t \le 4\pi$.

(a) At time $t = 3.5$, which direction is the particle moving? Find the speed of the particle at this time.

- $\begin{cases} x'(t) = t(-\sin t) + \cos t \\ y'(t) = t\cdot\cos t + \sin t \end{cases}$ $\begin{cases} x'(3.5) \approx 0.291 \\ y'(3.5) \approx -3.63 \end{cases}$ particle moves to the right and down at this time (↘)



t=3.5 particle moves down and right w/ speed of 3.64.

- Speed $= \sqrt{[x'(3.5)]^2 + [y'(3.5)]^2} = \cdots \approx 3.64$

(b) Find the average speed and the arc length of the particle on $[0, 4]$. What do you notice?

Avg speed $= \frac{1}{4-0}\int_0^4 \text{speed} = \frac{1}{4}\int_0^4 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \frac{1}{4}\int_0^4 \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t}$

expand + simplify

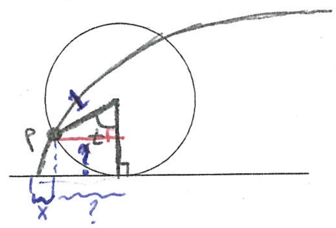
$= \frac{1}{4}\int_0^4 \sqrt{1+t^2}\,dt \approx 2.323$

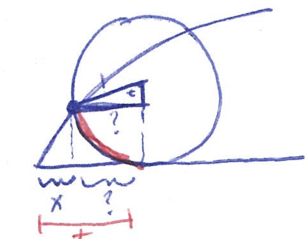
Recall: * Avg. value of blah on $[a,b]$ is $\frac{1}{b-a}\int_a^b \text{blah}$

Arc length $= \int_0^4 \text{speed} = \int_0^4 \sqrt{1+t^2}\,dt \approx 9.294$

5. A **cycloid** is a path traced by a point on the edge of a wheel.

(a) Find parametric equations for the cycloid generated by the wheel of radius 1 shown. Suppose $t$ is measured in radians.




Better picture...

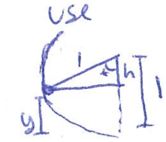
Now,

$X + ? = t$

$X = t - ?$

$X = t - \sin t$

To find y. use




$h = \cos t$

$y = 1 - h = 1 - \cos t$

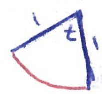

$\frac{?}{1} = \sin(t)$

$? = \sin(t)$

Other

X+? is the same as the arc of the circle described by angle t (see red on picture)

$$\boxed{\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}}$$




the red distance is proportional to t out of $2\pi$

$\frac{t}{2\pi} = \frac{d}{2\pi \cdot 1} \Rightarrow d = t$

(b) For what value(s) of $t$ is the tangent line horizontal?

$0 = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ means $y'(t) = 0$

$0 = \frac{d}{dt}(1 - \cos t)$

$0 = \sin t$

$t = 0, \pi, 2\pi, \ldots$ so $\boxed{t = \pi n}$ for whole #s $n$.

(c) For what value(s) of $t$ is the point stopped?

Point stopped means speed $= 0$.

Speed $= \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{1 - 2\cos t + 1}$

$= \sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{1 - \cos t}$.

$0 = \text{speed} = \sqrt{2}\sqrt{1-\cos t}$

$1 - \cos t = 0$

$\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi, \ldots$ so $\boxed{t = 2\pi n}$, where $n$ is a whole #