Math 2300-013: Integration By Parts

Compute the following integrals using integration by parts:

1. $\int x \cos x \, dx$ $= x\sin x - \int \sin x = x\sin x + \cos x + C$

$$\begin{bmatrix} u = x & dv = \cos x\, dx \\ du = dx & v = \sin x \end{bmatrix}$$

2. $\int_0^2 xe^x \, dx$ $= xe^x\Big|_0^2 - \int_0^2 e^x dx = (2e^2 - 0) - e^x\Big|_0^2$

$$\begin{bmatrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{bmatrix}$$

$= 2e^2 - (e^2 - 1)$

$= 2e^2 - e^2 + 1$

$= e^2 + 1$

3. $\int_4^9 \frac{\ln(y)}{\sqrt{y}} \, dy$ $= \int_4^9 \ln(y) \cdot y^{-1/2} dy = 2y^{1/2}\ln(y)\Big|_4^9 - \int_4^9 \frac{2y^{1/2}}{y} dy$

$$\begin{bmatrix} u = \ln(y) & dv = y^{-1/2} dy \\ du = \frac{1}{y} dy & v = 2y^{1/2} \end{bmatrix}$$

$= 2\sqrt{y}\ln(y)\Big|_4^9 - \int_4^9 2y^{-1/2} dy$

$= [2\cdot 3\ln(9) - 2\cdot 2\ln(4)] - 4y^{1/2}\Big|_4^9$

$= [6\ln(9) - 4\ln(4)] - [4\cdot 3 - 4\cdot 2]$

$= 6\ln(9) - 4\ln(4) - 4$

4. $\int \theta^2 \sin(3\theta)\, d\theta = -\frac{1}{3}\theta^2\cos(3\theta) + \int \frac{2}{3}\theta\cos(3\theta)\,d\theta$

$\begin{bmatrix} u=\theta^2 & dv=\sin(3\theta)d\theta \\ du=2\theta d\theta & v=-\frac{1}{3}\cos(3\theta) \end{bmatrix}$

$= -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{9}\theta\sin(3\theta) - \int \frac{2}{9}\sin(3\theta)\,d\theta$

$= -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{9}\theta\sin(3\theta) + \frac{2}{9}\cdot\frac{1}{3}\cos(3\theta) + C$

$\begin{bmatrix} u=\frac{2}{3}\theta & dv=\cos(3\theta)\,d\theta \\ du=\frac{2}{3}d\theta & v=\frac{1}{3}\sin(3\theta) \end{bmatrix}$

$= -\frac{1}{3}\theta^2\cos(3\theta) + \frac{2}{9}\theta\sin(3\theta) + \frac{2}{27}\cos(3\theta) + C$

5. $\int \arctan x\, dx = x\arctan x - \int \frac{x}{x^2+1}\,dx$

$\begin{bmatrix} u=\arctan x & dv=dx \\ du=\frac{1}{x^2+1}dx & v=x \end{bmatrix}$

$= x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du$

$= x\arctan x - \frac{1}{2}\ln|u| + C$

$= x\arctan x - \frac{1}{2}\ln|x^2+1| + C$

$\left\{ \begin{array}{l} \underline{\text{u-sub:}} \\ u=x^2+1 \\ du=2x\,dx \\ \frac{1}{2}du=x\,dx \end{array} \right\}$

6. $\int \ln x\, dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx$

$= x\ln x - x + C$

$\begin{bmatrix} u=\ln x & dv=dx \\ du=\frac{1}{x}dx & v=x \end{bmatrix}$

7. $\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx = x(\ln x)^2 - 2\int \ln x\,dx$

$\left[\begin{array}{ll} u=(\ln x)^2 & dv = dx \\ du = 2\ln x\cdot\frac{1}{x}\,dx & v = x \end{array}\right]$

$= x(\ln x)^2 - 2\left[x\ln x - x\right] + C$ see #6

$= x(\ln x)^2 - 2x\ln x + 2x + C$

8. $\int_0^1 \frac{x+1}{e^x}\,dx = \int_0^1 (x+1)e^{-x}\,dx = -(x+1)e^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx$

$\left[\begin{array}{ll} u=(x+1) & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{array}\right]$

$= \frac{-(x+1)}{e^x}\Big|_0^1 + -e^{-x}\Big|_0^1$

$= \left[\frac{-2}{e} + \frac{1}{1}\right] + \left[\frac{-1}{e} + 1\right]$

$= 2 - \frac{3}{e}$

9. $\int e^t\cos t\,dt = e^t\sin t - \int e^t\sin t\,dt = e^t\sin t - \left[-e^t\cos t + \int e^t\cos t\,dt\right]$

$\left[\begin{array}{ll} u=e^t & dv = \cos t\,dt \\ du = e^t dt & v = \sin t \end{array}\right]$

$= e^t\sin t + e^t\cos t - \int e^t\cos t\,dt.$

$\left[\begin{array}{ll} u=e^t & dv = \sin t\,dt \\ du = e^t dt & v = -\cos t \end{array}\right]$

Now,

$\int e^t\cos t\,dt = e^t\sin t + e^t\cos t - \int e^t\cos t\,dt$

So,

$2\int e^t\cos t\,dt = e^t\sin t + e^t\cos t$

$\int e^t\cos t\,dt = \frac{1}{2}\left[e^t\sin t + e^t\cos t\right] + C$

Note: Can also do

$u = \cos t \quad dv = e^t dt$

$du = -\sin t\,dt \quad v = e^t$

10. $\int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$$\begin{bmatrix} u = \sec x & dv = \sec^2 x\, dx \\ du = \sec x \tan x\, dx & v = \tan x \end{bmatrix}$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$
$$= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

By the "Boomerang"...

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$
$$\int \sec^3 x\, dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\, dx$$
$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$
$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$
$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \tfrac{1}{u}\, du$$
$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C$$

u-sub:
$$\begin{bmatrix} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x)\, dx \end{bmatrix}$$

11. $\displaystyle\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$

$$\begin{bmatrix} u = \arctan\left(\frac{1}{x}\right) & dv = dx \\ du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot \frac{d}{dx}\left(\frac{1}{x}\right) & v = x \\ = \frac{1}{1+\frac{1}{x^2}}\cdot \frac{-1}{x^2} & \\ = \frac{-1}{x^2+1} & \end{bmatrix}$$

$$= x\arctan\left(\tfrac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\, dx$$
$$= x\arctan\left(\tfrac{1}{x}\right)\Big|_1^{\sqrt{3}} + \frac{1}{2}\int_{u=2}^{u=4} \frac{1}{u}\, du$$

u-sub:
$$\begin{bmatrix} u = x^2+1 \\ du = 2x\, dx \\ \tfrac{1}{2} du = x\, dx \end{bmatrix}$$

$$= \left[\sqrt{3}\arctan\left(\tfrac{1}{\sqrt{3}}\right) - \arctan(1)\right] + \left[\tfrac{1}{2}\ln|u|\right]_2^4$$
$$= \left(\sqrt{3}\cdot\frac{\pi}{6} - \frac{\pi}{4}\right) + \left(\tfrac{1}{2}\ln(4) - \tfrac{1}{2}\ln(2)\right)$$
$$= \frac{\sqrt{3}\pi}{6} - \frac{3\pi}{12} + \frac{1}{2}\left[\ln\left(\tfrac{4}{2}\right)\right]$$
$$= \frac{2\sqrt{3}\pi - 3\pi}{12} + \frac{1}{2}\ln(2)$$
$$= \frac{\pi}{12}\left(2\sqrt{3}-3\right) + \ln\left(\sqrt{2}\right)$$

Note:
$$\tan\left(\tfrac{\pi}{6}\right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$
$$\tan\left(\tfrac{\pi}{4}\right) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$