Solutions

§8.7: Taylor's Inequality

(Created by Faan Tone Liu)

Key Points:

- **Goal:** Find the size of the error when using a Taylor Polynomial to estimate a function. The error (remainder) is $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is the nth degree Taylor polynomial of f.
- The size of the error is influenced by the following:
  - the degree n of the Taylor Polynomial
  - the distance between x and the point a around which the Taylor Series is centered
  - the size of $\left|f^{(n+1)}(x)\right|$ in the interval between x and a.
- **Taylor's Inequality:** If $f^{(n+1)}$ is continuous and $\left|f^{(n+1)}(x)\right| \leq M$ between x and a, then the remainder $R_n(x)$ satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

- Procedure: Find M and fill in a. Then $R_n(x)$, n and x interact. You'll typically be given two of them and have to find the third.

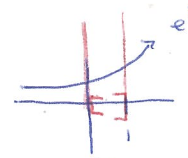
Examples:

1. Use the 6th degree Tayolor Polynomial for e^x, centered at $a = 0$ to estimate e. How accurate is your estimate guaranteed to be?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

estimate for e

$$e \approx \sum_{n=0}^{6} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.718067$$



Error

Find M, a bound on $|f^{(n+1)}(x)|$ for the interval $[0,1] \leftarrow \begin{cases} a=0 \\ x=1 \end{cases}$

$|f^{(7)}(x)| = |e^x| \leq e$ on $[0,1]$, $e < 3$,

e^x bdd by $e^1 = e$ (e^x increasing)

$$|R_6(1)| \leq \frac{M}{7!}|1-0|^7 < \boxed{\frac{3}{7!}\cdot 1^7} \approx \boxed{0.0005952}$$

2. How accurate is the 3rd degree Taylor Polynomial centered at 0 guaranteed to be in estimating $\sin(9°)$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$9° = \frac{9\cdot\pi}{180} \text{ rad} = \frac{\pi}{20} \text{ rad}$$

$$\sin(9°) = \sin\left(\frac{\pi}{20} \text{ rad}\right) \approx T_3\left(\frac{\pi}{20}\right) = \frac{\pi}{20} - \frac{(\pi/20)^3}{6} \approx 0.15643367$$

estimate for $\sin(9°)$

Error

4th derivative

Find M: $|f^{(3+1)}(x)| = |\sin x| \leq 1$

$$|R_3(9°)| \leq \frac{M}{4!}\left|\underset{x}{\frac{\pi}{20}} - \underset{a}{0}\right|^4 \leq \boxed{\frac{1}{4!}\left(\frac{\pi}{20}\right)^4 \approx 0.00002537}$$

3. Show that the Taylor Series for $\cos(x)$ centered at $a = 0$ converges to $\cos(x)$.

By Taylor's Inequality, the remainder for any x is

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-0|^{n+1}$$

Can choose $M=1$ since $f^{(n+1)}(x)$ is $\pm\sin(x)$ or $\pm\cos x$ for all n.

$$|R_n(x)| \leq \frac{1}{(n+1)!}\cdot|x|^n$$

$$0 \leq \lim_{n\to\infty}|R_n(x)| \leq \lim_{n\to\infty}\frac{|x|^n}{(n+1)!} = 0.$$

This shows that for any x, the Remainder between $T_n(x)$ and $\cos(x)$ goes to 0

4. How large should n be to estimate e to within six decimal places (0.0000005) using a Taylor Polynomial centered at $x = 0$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $a=0$, to approximate e, plug in $x=1$

Remainder: $|R_n(1)| \leq \frac{M}{(n+1)!}|1-0|^{n+1}$

Since $|f^{(n+1)}(x)| = |e^x| \leq e < 3$ on $[0,1]$ (see #1), we can pick $M=3$.

Now, we want $|R_n(1)| \leq \frac{3}{(n+1)!}1^{n+1} \leq 0.0000005$

$$\frac{3}{(n+1)!} \leq 0.0000005$$

[$n=10$ works by trial and error.]

5. Use a 5th degree Taylor Polynomial to estimate $\ln(1.1)$. How accurate is your estimate?

$f(x) = \ln(x+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ from previous worksheet $[a=0]$

we want ~~to~~ estimate $\ln(1.1) = f(1)$, so $x = 0.1$

$|f^{(n+1)}(x)| = \left|(-1)^{n+2}(n)!\frac{1}{(1+x)^{n+1}}\right|$ by #5 on prev. worksheet

$= \frac{n!}{|1+x|^{n+1}}$ this is decreasing on the interval between $a=0$ and $x=0.1$, so largest in size at 0.

$\leq \frac{n!}{|1+0|^{n+1}}$

$= n!$

$= M$

$$|R_5(0.1)| \leq \frac{M}{6!}|0.1-0|^6 \leq \frac{5!}{6!}(0.1)^6 = \frac{1}{5}\cdot\frac{1}{10^6}$$

$\boxed{\text{Our Estimate is w/in } \frac{1}{5}\cdot\frac{1}{10^6} \text{ of true value}}$