§8.3: Separable Differential Equations

Key Points:

- A **separable differential equation** is a differential equation that can be written in the form
$$f(y) \cdot \frac{dy}{dx} = g(x).$$
- To solve a separable differential equation:
  1. Separate the variables
  2. Integrate both sides (Remember $+C$!!!)
  3. Solve for y (if possible)
  4. Use the initial condition to find C.
- Other notes:

I should have given you more room on page 3!

Examples:

1. Solve the differential equation $\frac{dy}{dx} = -2y$ if $y(0) = 1$.

$$\frac{dy}{dx} = -2y$$
$$\frac{1}{y}\,dy = -2\,dx$$
$$\int \frac{1}{y}\,dy = \int -2\,dx$$

$$\ln|y| = -2x + C$$
$$|y| = e^{-2x+C}$$

Solve for C:
$$|1| = e^{-2\cdot 0 + C}$$
$$1 = e^{C}$$
$$C = 0$$

$$|y| = e^{-2x}$$
$$\boxed{y = e^{-2x}}$$

since in initial condition, $y = 1 \geq 0$.

2. Solve the differential equation $\frac{dx}{dt} + x = 1$ if $x(1) = 0.1$.

$$\frac{dx}{dt} = 1 - x$$
$$\frac{1}{1-x}\,dx = dt$$
$$\int \frac{1}{1-x}\,dx = \int dt$$
$$-\ln|1-x| = t + C$$
$$\ln|1-x| = -(t+C)$$

Remember chain rule! (or do u-sub $u = 1-x$)

$$|1-x| = e^{-(t+c)}$$

Solve for C:
$$|1-0.1| = e^{-(1+C)}$$
$$0.9 = e^{-1-C}$$
$$\ln(0.9) = -1 - C$$
$$C = -\ln(0.9) - 1$$

$$|1-x| = e^{-t+\ln(0.9)+1}$$
$$1 - x = e^{1+\ln(0.9)-t}$$
$$\boxed{x = 1 - e^{1+\ln(0.9)-t}}$$

Can remove abs. value because initial condition $(1, 0.1)$ (t, x) yields $1 - x = 1 - 0.1 = 0.9 \geq 0$.

3. Solve the differential equation $\frac{du}{dt} = u + ut^2$ if $u(0) = 5$.

$$\frac{du}{dt} = u(1+t^2)$$
$$\int \frac{1}{u}\,du = \int (1+t^2)\,dt$$
$$\ln|u| = t + \tfrac{1}{3}t^3 + C$$
$$|u| = e^{t+\frac{1}{3}t^3+C}$$

Solve for C:
$$|5| = e^{0+\frac{1}{3}\cdot 0^3 + C}$$
$$5 = e^C$$
$$C = \ln(5)$$

$$|u| = e^{t+\frac{1}{3}t^3+\ln(5)}$$

Remove abs values because they aren't needed for initial condition $(|5| \geq 0.)$

$$\boxed{u = e^{t+\frac{1}{3}t^3+\ln(5)}}$$

4. Solve the differential equation $\frac{dy}{dx} = xe^y$ if $y(0) = 0$.

$$\frac{dy}{dx} = xe^y$$
$$\int e^{-y}\,dy = \int x\,dx$$
$$-e^{-y} = \tfrac{1}{2}x^2 + C$$
$$e^{-y} = -\tfrac{1}{2}x^2 - C$$
$$-y = \ln(-\tfrac{1}{2}x^2 - C)$$
$$y = -\ln(-\tfrac{1}{2}x^2 - C)$$

Solve for C:
$$0 = -\ln(-\tfrac{1}{2}\cdot 0^2 - C)$$
$$0 = -\ln(-C)$$
$$e^0 = -C$$
$$-1 = C$$

$$\boxed{y = -\ln(1 - \tfrac{1}{2}x^2)}$$

5. Solve the differential equation $\frac{ds}{d\theta} = -s^2\tan\theta$ if ~~$y(0) = 0$.~~ $\boxed{s(0) = 2}$ oops!

$$\frac{ds}{d\theta} = -s^2\tan\theta$$
$$-\frac{1}{s^2}\,ds = \tan\theta\,d\theta$$
$$\int -s^{-2}\,ds = \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$$s^{-1} = \int -\frac{1}{u}\,du$$
$$\frac{1}{s} = -\ln|u| + C$$
$$\frac{1}{s} = -\ln|\cos\theta| + C$$

Solve for C:
$$\frac{1}{2} = -\ln|\cos(0)| + C$$
$$\frac{1}{2} = -\ln|1| + C$$
$$\frac{1}{2} = C$$

$$\frac{1}{s} = -\ln|\cos\theta| + \frac{1}{2}$$

$$\boxed{s = \frac{1}{\frac{1}{2} - \ln|\cos\theta|}}$$

6. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enteres the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

We want an equation $S(t)$ for amount of salt at time t.
First, let's understand the rates we are given:

$$\frac{dS}{dt} = \text{Change from brine entering} - \text{Change from brine leaving}$$

[rate = rate in − rate out]

Change in salt over time

current concentration of salt $\frac{kg}{L} \cdot \frac{L}{m}$

$$\frac{dS}{dt} = \underbrace{(0.03)\cdot 25}_{\frac{kg}{L}\cdot\frac{L}{min}} - \frac{S}{5000}\cdot 25 = \left(0.03 - \frac{S}{5000}\right)25 \text{ in } \frac{kg}{min}.$$

$$\frac{1}{5000}\cdot \frac{1}{0.03 - \frac{1}{5000}S}\,dS = 25\,dt \cdot \frac{1}{5000}$$

$$\int \frac{1}{150 - S}\,dS = \int 0.005\,dt$$

$$-\ln|150 - S| = 0.005t + C$$

$$-\ln|150 - S| = 0.005t - 5.01$$

Solve for C: We know $S(0) = \frac{20}{5000}$

$$-\ln|150 - 0.004| = 0 + C$$

$$C \approx -5.01$$

We want $S(30)$: Can solve for S when $t = 30$:

$$-\ln|150 - S| = 0.005 \cdot 30 - 5.01$$

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$$\boxed{S = 20.897 \text{ kg}}$$

7. Find an equation of the curve that passes through the point $(0, 1)$ and whose slope at (x,y) is xy.

$$\frac{dy}{dx} = xy$$

Slope

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{\frac{1}{2}x^2 + C}$$

$$|y| = e^{\frac{1}{2}x^2}$$

Solve for C: We know that $y(0) = 1$, so

$$|1| = e^{\frac{1}{2}\cdot 0^2 + C}$$

$$1 = e^C$$

$$C = 0$$

$$\boxed{y = e^{\frac{1}{2}x^2}}$$

Can remove abs. values because in the initial condition, $y = 1 \geq 0$.