The Comparison Test
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ converges and $a_n \leq b_n$, then $\sum a_n$ also converges.
- If $\sum b_n$ diverges and $a_n \geq b_n$, then $\sum a_n$ also diverges.

The Limit Comparison Test
Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Each of the following series can be proven to converge or diverge by comparing to a known series. For some of these series you can compare the term-size to get a result. For others, simple comparison doesn't work quite right and instead you must use the Limit Comparison Test. For each of the following, determine what known series to compare to, and which test should be used. Use that test to show convergence or divergence of the series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2^n}$

   Solution: I'll use the Term-size Comparison Test. Let $a_n = \dfrac{\arctan n}{2^n}$ and $b_n = \dfrac{2}{2^n}$, both are positive.

   Since $\arctan n \leq 2$ for all n, $\dfrac{\arctan n}{2^n} \leq \dfrac{2}{2^n}$ for all $n > 0$.

   $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{2^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a convergent geometric series ($r = \frac{1}{2} < 1$).

   Therefore, by the Term-size Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2^n}$ also converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$

   Solution: Term-size Comparison Test. Let $a_n = \dfrac{1}{\sqrt{n}}$ and $b_n = \dfrac{\arctan n}{\sqrt{n}}$, both sequences are positive.

   $\dfrac{1}{\sqrt{n}} \leq \dfrac{\arctan n}{\sqrt{n}}$ for $n > 1$.

   We saw previously that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2} < 1$).

   Therefore, by the Term-size Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$ also diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$

Solution: Term-size Comparison Test. Let $a_n = \dfrac{1}{e^n + n^2}$ and $b_n = \dfrac{1}{n^2}$ (both positive).

$\dfrac{1}{e^n + n^2} \le \dfrac{1}{n^2}$ for any $n > 0$.

We saw previously that $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$).

Therefore, by the Term-size Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$ also converges.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$

Solution: Limit Comparison Test. Let $a_n = \dfrac{1}{e^n}$ and $b_n = \dfrac{1}{e^n - n^2}$ (both positive).

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^n - n^2}{e^n} = 1 - \lim_{n\to\infty} \frac{n^2}{e^n} = 1$ (By L'Hopital's Rule). The limit is finite and not zero, so the Limit Comparison Test applies.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a convergent geometric series ($r = \frac{1}{e} < 1$).

Therefore, by the Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$ also converges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n - 2}{2n^3 + 5}$

Solution: Term-size Comparison Test. Let $a_n = \dfrac{3n - 2}{2n^3 + 5}$ and $b_n = \dfrac{3n}{n^3}$ (both positive).

$\dfrac{3n - 2}{2n^3 + 5} \le \dfrac{3n}{n^3}$ for any $n > 0$.

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3n}{n^3} = 3\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2$, which is greater than 1).

Therefore, by the Term-size Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{3n - 2}{2n^3 + 5}$ also converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n + 5}{n^3 - 3n + 6}$

Solution: Limit Comparison Test. Let $a_n = \dfrac{n^2 - n + 5}{n^3 - 3n + 6}$ and $b_n = \dfrac{1}{n}$ (both positive).

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3 - n^2 + 5}{n^3 - 3n + 6} = \lim_{n\to\infty} \frac{1 - \frac{1}{n} + \frac{5}{n^3}}{1 - \frac{3}{n^2} + \frac{6}{n^3}} = 1$. The limit is finite and not zero, so the Limit Comparison Test applies.

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1$).

Therefore, by the Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n + 5}{n^3 - 3n + 6}$ also diverges.