1. Let's start with the **sequence** $a_n = \frac{1}{n}$. Does this **sequence** converge or diverge? Explain.

   Solution: To determine if a sequence converges, we just take a limit: $\lim_{n\to\infty} \frac{1}{n} = 0$. The sequence converges to 0.

2. Now consider this infinite **series** (called the harmonic series):

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

   (a) Write this series in summation notation.

   Solution: $\sum_{n=1}^{\infty} \frac{1}{n}$

   (b) Now consider this series numerically. Calculate the following partial sums:

$$\sum_{n=1}^{1} \frac{1}{n} = 1$$

$$\sum_{n=1}^{2} \frac{1}{n} = \frac{3}{2} = 1.5$$

$$\sum_{n=1}^{3} \frac{1}{n} \approx 1.833$$

$$\sum_{n=1}^{4} \frac{1}{n} \approx 2.083$$

$$\sum_{n=1}^{10} \frac{1}{n} \approx 2.929$$

$$\sum_{n=1}^{100} \frac{1}{n} \approx 5.187$$

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.48$$

   (c) Recall that convergence of an infinite series is determined by taking the limit of the partial sums. In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n}$$

   If the limit is finite, we say the infinite series converges, otherwise we say it diverges. Using what you calculated in part(b) of this problem, make a conjecture about whether or not the harmonic series converges.

   Solution: The numbers don't appear to be "settling down" yet, which makes me suspicious that it won't converge. But the numbers are growing slowly, so I'm suspicious that the total won't grow without bound (diverge to infinity). I'm going to guess that it diverges, but it's really just a guess.

3. The next part of the project introduces the concept of the Integral Test to show a series diverges.

   (a) Every series can be depicted graphically. Write down a sum that gives the area of the shaded region below. How does this sum relate to the harmonic series?
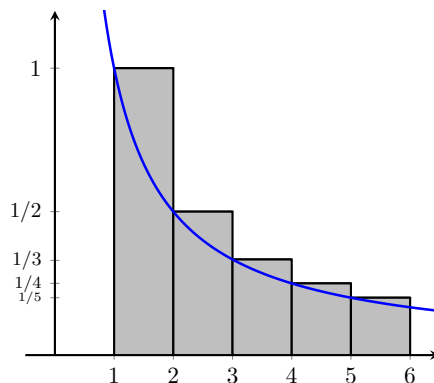
   Solution: The width of each rectangle is 1 unit, the heights are 1, $\frac{1}{2}$, $\frac{1}{3}$, etc. So the total area is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}...$, the harmonic series.

   (b) The advantage of representing a series this way is that it can be compared to an improper integral. On the above graph, *carefully* draw the function $f(x) = \frac{1}{x}$.

   (c) How does the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ compare to the improper integral $\int_{1}^{\infty} \frac{1}{x} \, dx$?

   Solution: The improper integral is contained in the region that represents the harmonic series. The sum of the harmonic series is bigger than the area representing the improper integral.

   (d) Does the improper integral $\int_{1}^{\infty} \frac{1}{x} \, dx$ converge or diverge? Calculate it, as a review of improper integrals.

   Solution:

   $$\int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx$$
   $$= \lim_{b \to \infty} \ln|x|\Big|_{1}^{b} = \lim_{b \to \infty} \left(\ln|b| - \ln|1| \right)$$
 $$= \lim_{b \to \infty} \ln|b| = \infty$$

   It diverges.

   (e) What can you conclude about the convergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$?

   Solution: Since the series is larger than the integral, and the integral diverges, the series must also diverge.

2

4. In the previous problem we compared an infinite series to an improper integral to show divergence of the infinite series. By shifting to the left where we draw the rectangles, we can compare an infinite series to an improper integral to show convergence of the series.

(a) Write down a sum that gives the area of the shaded region below. How does this sum relate to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?
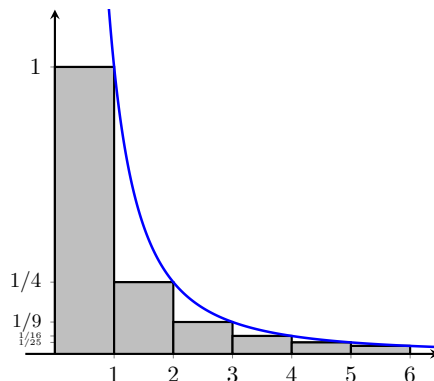
Solution: The width of each rectangle is 1 unit, the heights are 1, $\frac{1}{4}$, $\frac{1}{9}$, etc. So the total area is $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}...$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(b) On the above graph, *carefully* draw the function $f(x) = \frac{1}{x^2}$.

(c) How does the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ compare to the improper integral $\int_{1}^{\infty} \frac{1}{x^2}\, dx$?

Solution: The improper integral contains the region that represents the series. The sum of the series is smaller than the area representing the improper integral.

(d) Does the improper integral $\int_{1}^{\infty} \frac{1}{x^2}\, dx$ converge or diverge? Calculate it, as a review of improper integrals.
Solution:

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-2}\, dx$$
$$= \lim_{b \to \infty} -x^{-1}\Big|_{1}^{b} = \lim_{b \to \infty} \left[\left(-b^{-1}\right) - \left(-(1)^{-1}\right)\right]$$
$$= \lim_{b \to \infty} \left[-\frac{1}{b} + 1\right] = 0 + 1 = 1$$

It converges.

(e) What can you conclude about the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$, and thus about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

> **Solution:**   The integral is larger than the first sum, and since the integral converges, the first sum must also converge. The second sum is only 1 larger than the first sum, and since the first sum converges, the second sum must also converge.

5. In problems 3 and 4 we compared infinite series to improper integrals in order to make conclusions about the convergence of divergence of the infinite series. Here is the general result:

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges. In other words:

- if we know $\int_1^{\infty} f(x)\, dx$ *converges*, then we know $\sum_{n=1}^{\infty} a_n$ _____converges_____.
- if we know $\int_1^{\infty} f(x)\, dx$ *diverges*, then we know $\sum_{n=1}^{\infty} a_n$ _____diverges_____.

---

6. Why do you think we need $f(x)$ to be decreasing? Think about what might go wrong with the geometrical argument if $f(x)$ isn't decreasing.

> **Solution:**   If $f(x)$ matches a_n at the integers, but isn't forced to be decreasing, then in-between the integers $f(x)$ could swoop above the rectangles whose areas represent $\sum a_n$. Then our argument that $\sum a_n$ is larger than the improper integral wouldn't hold. Consequently, in some cases $\int_1^{\infty} f(x)\, dx$ could diverge even if $\sum a_n$ converges.

7. Now we'll apply the Integral Test in an example. Determine whether $\sum_{n=1}^{\infty} \dfrac{1}{n^2 + 1}$ converges or diverges by following these steps:

$a_n = \dfrac{1}{n^2 + 1}$; so let $f(x) = $ _____$\dfrac{1}{x^2 + 1}$_____

$f(x)$ is decreasing because: (Hint: find $f'(x)$)

> **Solution:** $f'(x) = -2x(x^2 + 1)^{-2}$, which is negative. So $f(x)$ is decreasing.

Check that $f(x)$ is positive:

> **Solution:**   Yes, because x^2 is positive, $1 + x^2$ is positive, and $\dfrac{1}{1 + x^2}$ is positive.

We've fulfilled the hypotheses of the integral test, so we can conclude that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ and $\int_1^{\infty} \dfrac{1}{x^2+1}\,dx$ either both converge or both diverge.

Integrate to determine whether $\int_1^{\infty} \dfrac{1}{x^2+1}\,dx$ converges or diverges:

Solution:

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$$
$$= \lim_{b\to\infty} \arctan x \Big|_1^b$$
$$= \lim_{b\to\infty} (\arctan b - \arctan 1)$$
$$= \lim_{b\to\infty} \arctan b - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\int_1^{\infty} \dfrac{1}{x^2+1}\,dx$ _____converges_____, and therefore $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ also _____converges_____.

8. Use the integral test to determine whether $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$ converges or diverges.

Solution: Let $f(x) = \dfrac{1}{x\ln x}$. Since $n > 1$, both n and $\ln n$ are positive, and so $f(x)$ is positive.

$f'(x) = \dfrac{-(1+\ln n)}{(n\ln n)^2} < 0$, so $f(x)$ is decreasing. The integral test applies, so now we'll check for convergence of the improper integral.

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx$$
$$\text{(substitute } u = \ln x, \text{ and } du = \frac{1}{x}\,dx)$$
$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\,du$$
$$= \lim_{b\to\infty} \ln|u| \, \Big|_{\ln 2}^{\ln b}$$
$$= \lim_{b\to\infty} \ln\ln b - \ln\ln 2 = \infty$$

We conclude $\int_2^{\infty} \dfrac{1}{x\ln x}\,dx$ diverges, and thus $\int_1^{\infty} \dfrac{1}{x^2+1}\,dx$ diverges.