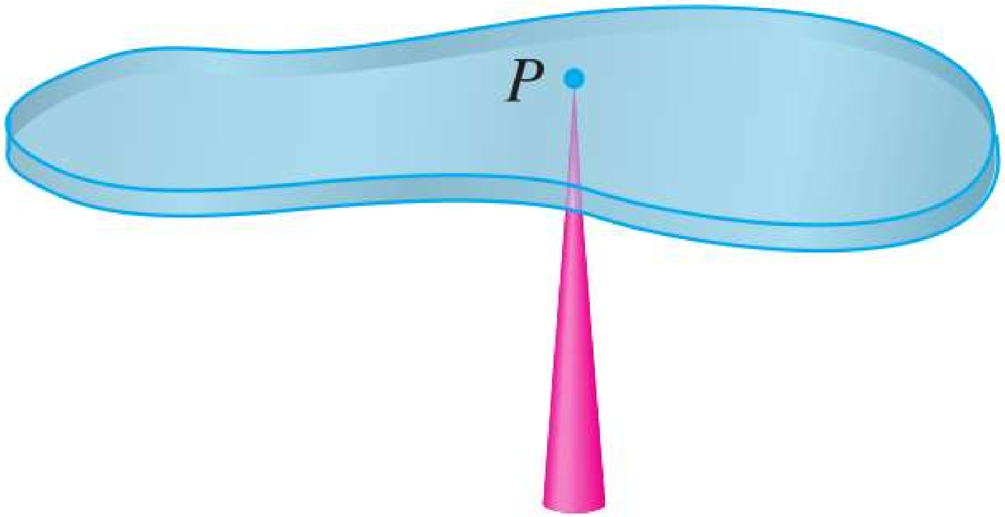
Daily Quiz

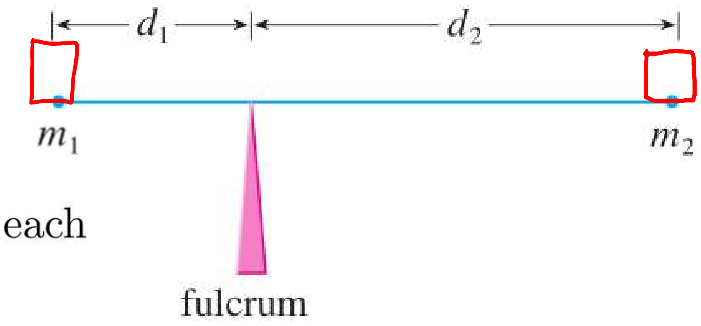
- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

6.6 Moment and Center of Mass

- Have you tried balancing a frisbee on your fingertips?
- Where would you put your finger?
- The center of mass is where the object would be balanced horizontally.
- It is also where the net torque from gravity would be 0.



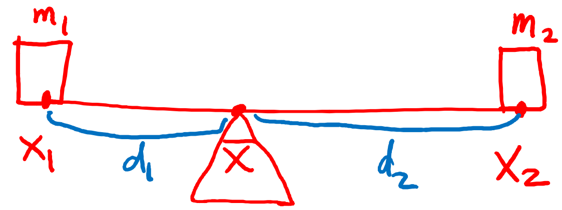
6.6 Moment and Center of Mass



- Consider a seesaw as pictured.
- If two people are sitting on the opposite ends of the **balanced** seesaw each with mass m_1 and m_2 respectively, who has more mass?
- Archimedes discovered that in order for the seesaw to be balanced, the moments need to be the same in magnitude:

$$M_1 = M_2$$
$$m_1 d_1 = m_2 d_2.$$

6.6 Moments and Centers of Mass



- Now suppose the two ends of the seesaw have x-coordinates x_1 and x_2.
- Let's compute the x-coordinate of the fulcrum, the center of mass $\bar{x}$.

$$m_1 d_1 = m_2 d_2$$

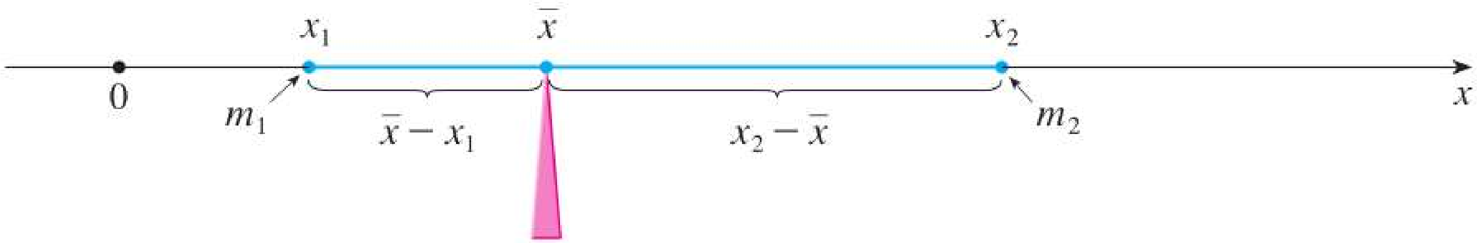
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

6.6 Moment and Center of Mass



- The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 with respect to a coordinate system.
- Note that x_1 and x_2 can be negative. This means that moments can be positive or negative depending on your choice of coordinate system.

https://phet.colorado.edu/en/simulation/balancing-act

6.6 Moment and Center of Mass

If we have a system of many particles with masses $m_1, m_2, \cdots, m_n$ located at the points $x_1, x_2, \cdots, x_n$ on the x-axis, then it can be shown similarly that the center of mass is located at

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\text{Sum of the moments}}{\text{Sum of the masses}}$$

6.6 Moment and Center of Mass (2-dim)

The center of mass is located at $(\overline{x}, \overline{y})$. Let m be the total mass, $m = m_1 + m_2 + \cdots + m_n$.

moment of the system about the *y*-axis

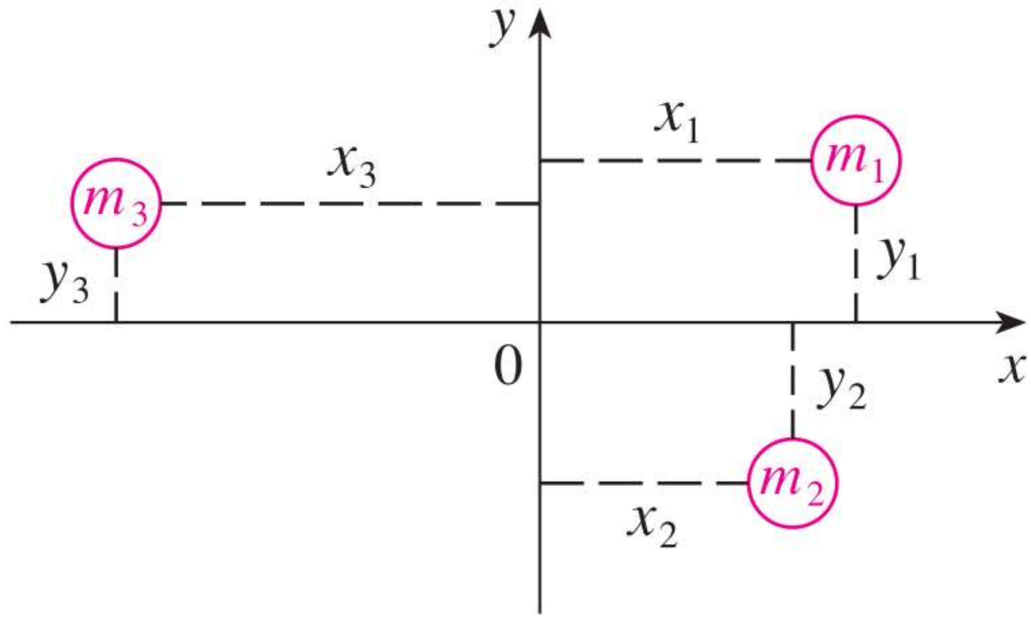
$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$$

$$\overline{x} = \frac{M_y}{m} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}$$

moment of the system about the *x*-axis

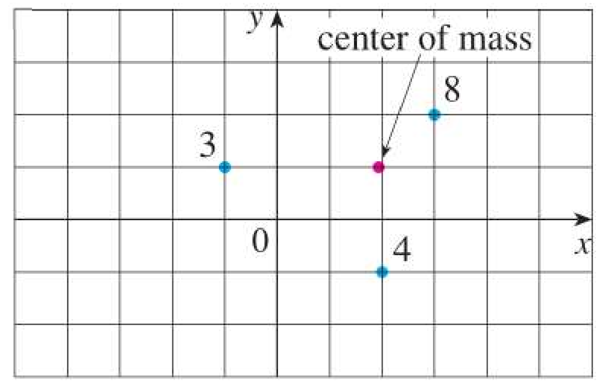
$$M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$$

$$\overline{y} = \frac{M_x}{m} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n}$$



6.6 Moment and Center of Mass (2-dim)

Find the moments and center of mass of the system of objects that have masses 3,4, and 8 at the points $(-1,1), (2,-1)$, and $(3,2)$.



$$\bar{x} = \frac{M_y}{m} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{-3+8+24}{3+4+8} = \frac{29}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{3-4+16}{3+4+8} = \frac{15}{15}$$

$M_y = 29$

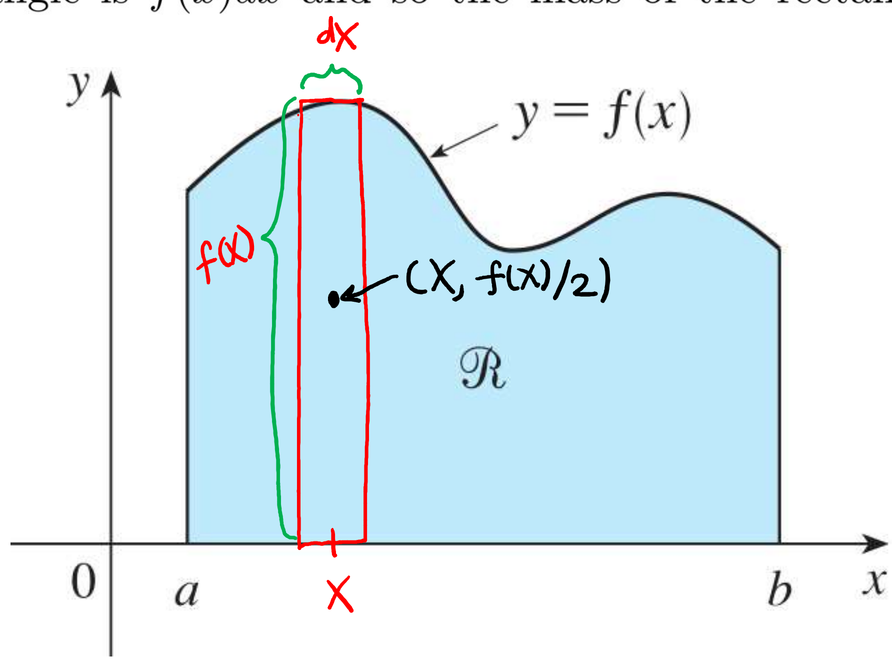
$M_x = 15$

The center of mass is located at $\left(\frac{29}{15}, 1\right)$

Suppose that a region of uniform density ρ is bounded by a function $y = f(x)$. How can we compute the center of mass of this region?

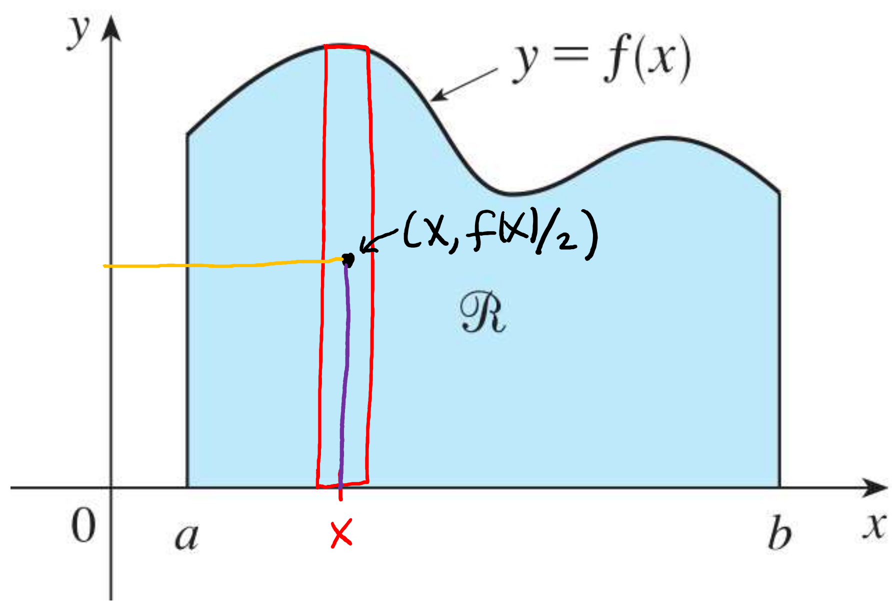
Suppose we take a rectangle of small width on the region $\mathscr{R}$. Intuitively, the center of mass of a rectangle is in the middle of the rectangle with coordinates $(x, f(x)/2)$. Now we can pretend that the entire mass of the rectangle is located at $(x, f(x)/2)$ and compute our moments with respect to the y and the x axis.

Firstly, the area of the rectangle is $f(x)dx$ and so the mass of the rectangle is $\rho f(x)dx$.

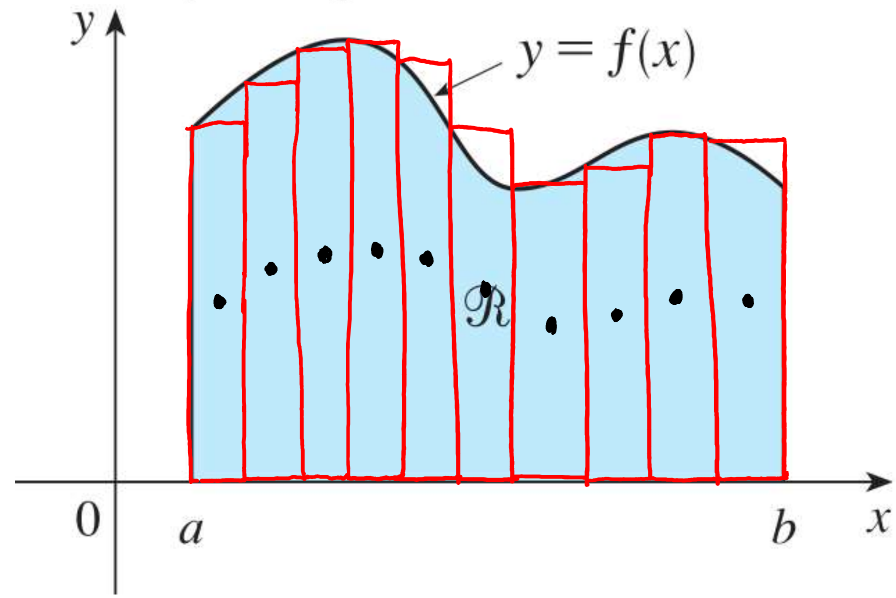


Let $\mathcal{M}_y$ denote the moment of the small rectangle about the y-axis and $\mathcal{M}_x$ denote the moment of the small rectangle about the x-axis. Then

$$\mathcal{M}_y = \text{mass} \cdot \text{distance to the y-axis} = \rho f(x)dx \cdot x$$
$$\mathcal{M}_x = \text{mass} \cdot \text{distance to the x-axis} = \rho f(x)dx \cdot f(x)/2$$



We have computed the moments of an arbitrary slice of rectangle on the region $\mathscr{R}$. Then the total moment of $\mathscr{R}$ about the y-axis is obtained by adding up slices' moments.



$$M_y = \int \mathcal{M}_y = \rho \int_a^b x f(x)\ dx$$

Similarly, the moment of $\mathscr{R}$ about the x-axis is

$$M_x = \int \mathcal{M}_x = \rho \int_a^b \frac{1}{2}[f(x)]^2\ dx$$

Now let's compute the mass of the region $\mathscr{R}$. Since the region has a uniform density, the mass of $\mathscr{R}$ is equal to the area of $\mathscr{R}$ times the density ρ.

$$m = \rho A = \rho \int_a^b f(x) dx.$$

Formulas

Moment about the y-axis:

$$M_y = \rho \int_a^b x f(x) dx$$

Moment about the x-axis:

$$M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx$$

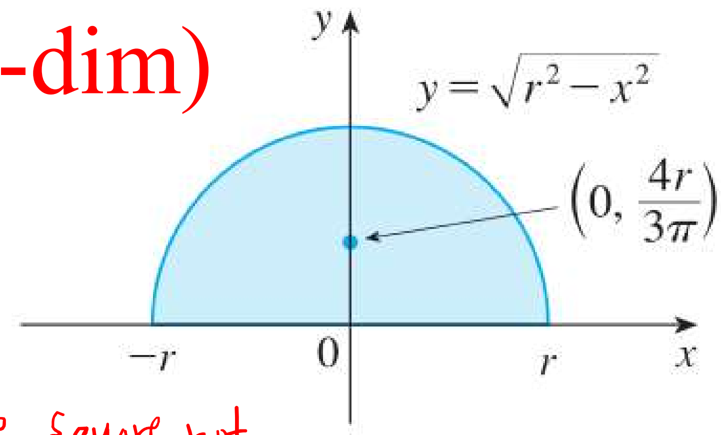
The center of mass of a region $\mathscr{R}$ is located at $(\overline{x}, \overline{y})$ and

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x) dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx$$

$$\overline{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x) dx]^2}{\rho \int_a^b f(x) dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2 dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2 dx$$

where $A = \displaystyle\int_a^b f(x) dx$.

6.6 Moment and Center of Mass (2-dim)



EXAMPLE 7 Find the center of mass of a semicircular plate of radius r.

Area is $\frac{1}{2}\pi r^2$.

Since the equation of a circle is $x^2+y^2=r^2$, the top half of the circle can be obtained from the positive square root when solving for y.

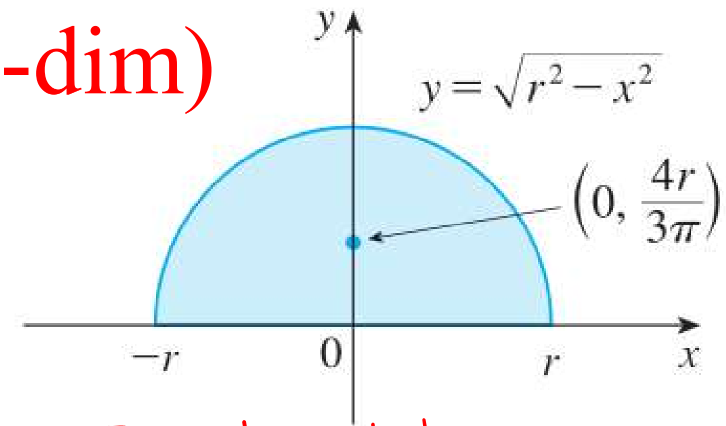
Top half: $y=\sqrt{r^2-x^2}$

Bottom half: $y=-\sqrt{r^2-x^2}$

Semicircle is top half. $f(x)=y=\sqrt{r^2-x^2}$

6.6 Moment and Center of Mass (2-dim)

EXAMPLE 7 Find the center of mass of a semicircular plate of radius r.



Using the formulas

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx = \frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx$$

Using u-sub

$u = r^2 - x^2$

$du = -2x\,dx$

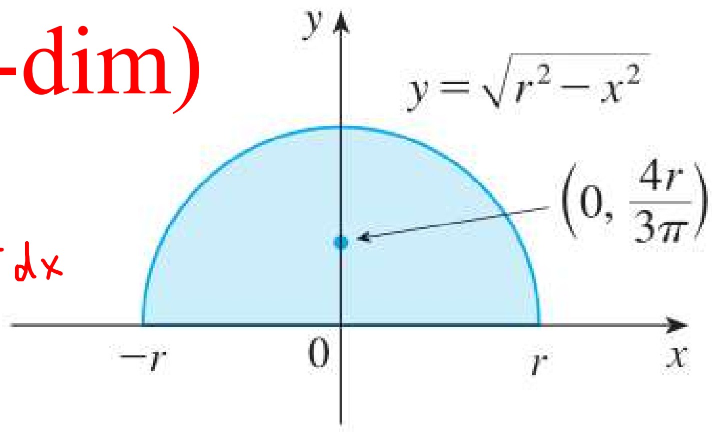
$$\frac{2}{\pi r^2}\int_0^0 \sqrt{u}\,\frac{du}{-2}$$

Observe that the new u-bounds are both 0 since $u = r^2 - r^2 = 0$ and $u = r^2 - (-r)^2 = 0$. Since an integral over an interval of no width is 0, the above integral is 0.

Hence $\frac{2}{\pi r^2}\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx = 0$. This makes intuitive sense since the object in question is symmetric across the y-axis.

$\bar{x} = 0$.

6.6 Moment and Center of Mass (2-dim)



EXAMPLE 7 Find the center of mass of a semicircular plate of radius r.

To compute $\bar{y}$, $\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{\frac{\pi r^2}{2}} \cdot \frac{1}{2}\int_{-r}^{r} r^2 - x^2\,dx$

$= \frac{1}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$

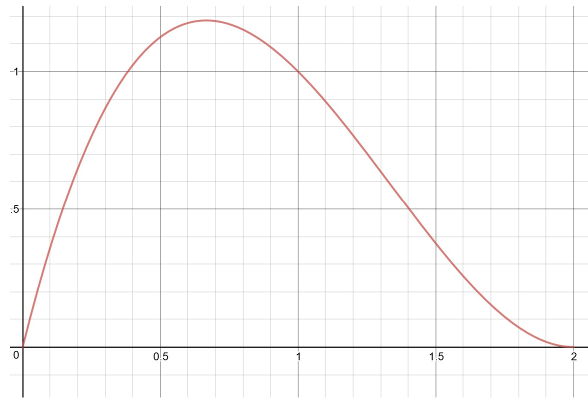
$= \frac{1}{\pi r^2}\,\frac{4}{3}r^3 = \frac{4r}{3\pi}.$

So $\bar{y} = \frac{4r}{3\pi}$.

center of mass $= (\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$

6.6 Moment and Center of Mass (2-dim)

Find the center of mass of a region bounded by $y = x(x-2)^2$ and the x-axis.



$$\text{Area} = \int_0^2 x(x-2)^2\,dx = \int_0^2 x(x^2-4x+4)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\right]_0^2$$

$$= \left[\left(\frac{16}{4} - \frac{4}{3}(8) + 8\right) - 0\right]$$

$$A = \frac{4}{3}$$

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^2(x-2)^2\,dx$$

$$= \frac{3}{4}\int_0^2 x^4 - 4x^3 + 4x^2\,dx$$

$$= \frac{3}{4}\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2$$

$$= \frac{3}{4}\left[\frac{32}{5} - 16 + \frac{32}{3}\right] = \frac{3}{4}\left[\frac{256}{15} - \frac{240}{15}\right] = \frac{4}{5}$$

6.6 Moment and Center of Mass (2-dim)

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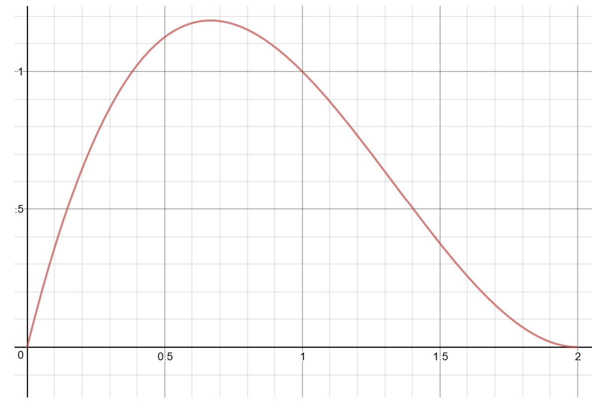

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx = \frac{3}{4}\cdot\frac{1}{2}\int_0^2 x^2(x-2)^4 dx$$

$$= \frac{3}{8}\int_0^2 x^2(x^4 + 4(-2)x^3 + 6(-2)^2x^2 + 4(-2)^3x + 16)\, dx$$

$$= \frac{3}{8}\int_0^2 x^6 - 8x^5 + 24x^4 - 32x^3 + 16x^2 dx$$

$$= \frac{3}{8}\left[\frac{x^7}{7} - \frac{8}{6}x^6 + \frac{24}{5}x^5 - \frac{32}{4}x^4 + \frac{16x^3}{3}\right]_0^2$$

$$= \frac{16}{35}$$

center of mass $= (\bar{x}, \bar{y})$

$$= \left(\frac{4}{5}, \frac{16}{35}\right)$$