Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

6.5 Average Value of a Function

For finitely many numbers $y_1, y_2, \cdots, y_n$, it is intuitive to compute the average value:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

Suppose we have a function $y = f(x)$, $a \leq x \leq b$. We want to find the average y-value as x varies from a to b. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then using the left endpoints, the average of the n values is

$$\frac{f(x_1) + \cdots + f(x_n)}{n}$$

But $n = (b-a)/\Delta x$. After substituting, we get

$$\begin{aligned}
\frac{f(x_1)+\cdots+f(x_n)}{n} &= \frac{f(x_1)+\cdots+f(x_n)}{\frac{b-a}{\Delta x}} \\
&= \frac{1}{b-a}[f(x_1)\Delta x + \cdots + f(x_n)\Delta x] \\
&= \frac{1}{b-a}\sum_{i=1}^{n} f(x_n)\Delta x
\end{aligned}$$

As n increases, Δx gets smaller so we would be averaging more numbers that are closer to each other. Take the limit as $n \to \infty$, then it will be as if **all** values of $f(x)$ are being averaged in the interval $[a,b]$. By the definition of an integral,

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(x_1)+\cdots+f(x_n)}{n} &= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_n)\Delta x \\
&= \frac{1}{b-a}\int_a^b f(x)\ dx
\end{aligned}$$

In conclusion, given a function $f(x)$, the **average value of** f on the interval $[a,b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Example. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution. Using the formula above,

$$\begin{aligned}
f_{\text{avg}} &= \frac{1}{b-a}\int_a^b f(x)\ dx \\
&= \frac{1}{2-(-1)}\int_{-1}^2 (1+x^2)\ dx \\
&= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^2 \\
&= \frac{1}{3}\left[\left(2+\frac{8}{3}\right) - \left(-1+\frac{-1}{3}\right)\right] \\
&= 2
\end{aligned}$$

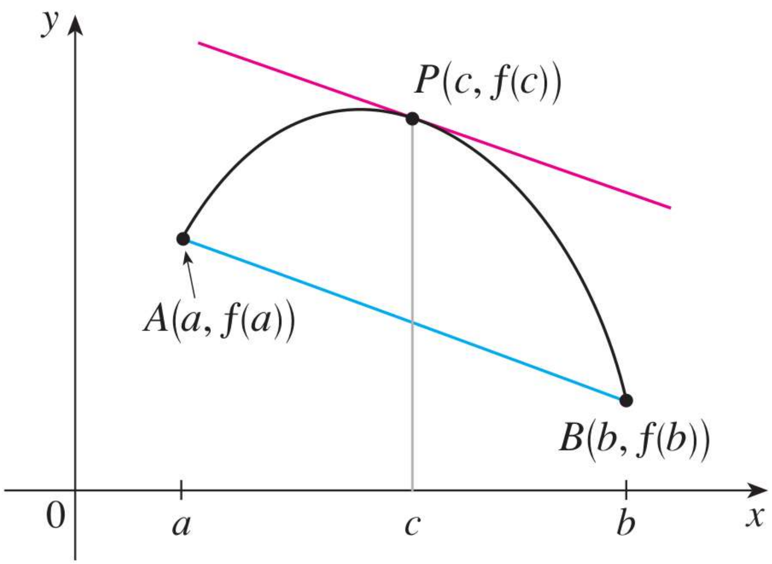
Mean Value Theorem. If f is a differentiable function on the interval $[a, b]$, then there exists a number c between a and b such that

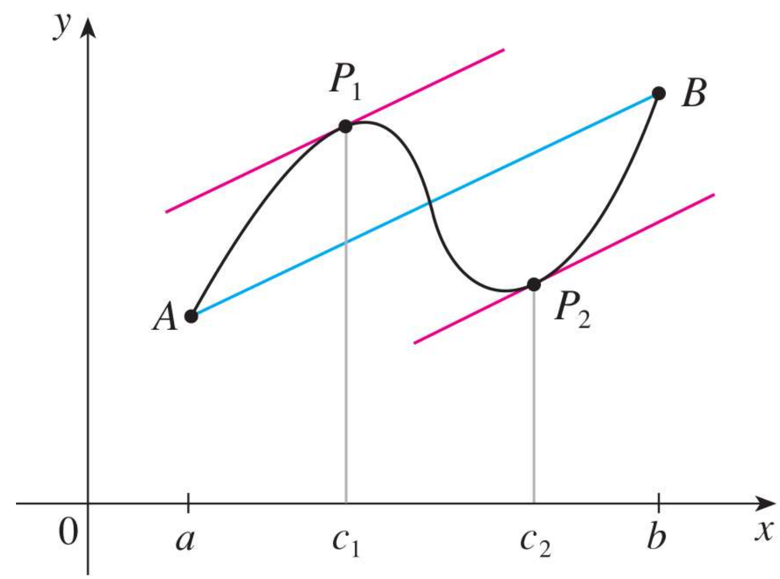
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

Note that this c may not be unique, as demonstrated by c_1 and c_2 on the second graph.



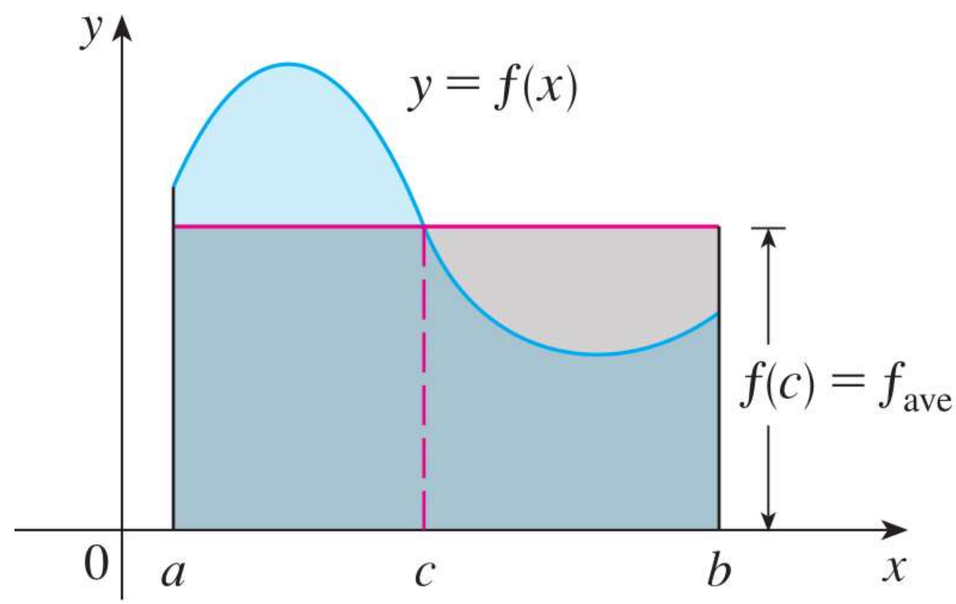


Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then ther exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

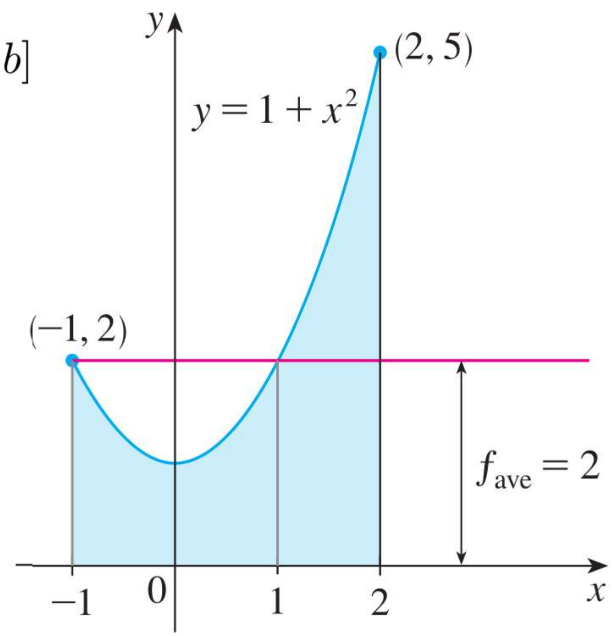
that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$



Geometric Interpretation. For positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

Example. Let's apply the MVT for Integrals on the previous example $f(x) = 1 + x^2$ over $[-1, 2]$ and explicitly locate c.



Recall that $f_{\text{avg}} = 2$. Then the MVT for Integrals says there is some $c \in [a, b]$ such that

$$f(c) = f_{\text{avg}} = 2, \qquad \int_{-1}^{2} (1 + x^2)\, dx = f(c)[2 - (-1)]$$

Since we are given the formula for $f(x)$, c is found by plugging it in to $f(x)$:

$$\begin{aligned} 2 &= f(c) = 1 + c^2 \\ 1 &= c^2 \\ c &= \pm 1 \end{aligned}$$

Since both ± 1 are inside the interval $[-1, 2]$, either value of c works for the MVT for Integrals.

Example. Let $s(t)$ be the displacement of a car at time t; let $v(t)$ be its velocity. In physics the average velocity of a particle moving in one dimension is given by the equation

$$\overline{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

In the previous slides, given **any** function f, we defined its average f_{avg} over an interval $[t_1, t_2]$. If we let $f(t) = v(t)$, does the average velocity in physics (which comes with real-world intuition) agree with the average value of a function in mathematics (which is defined without any relation to the physical world)?

Solution. Yes they agree;

$$\begin{aligned} v_{\text{avg}} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\ dt \\ &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t)\ dt \\ &= \frac{1}{t_2 - t_1} [s(t_2) - s(t_1)] \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \overline{v} \end{aligned}$$