Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

- Office hours updated: MW 3-4 PM
- MARC hour unchanged: F 1-2 PM

5.10 Comparing Improper Integrals

- If the integrals are extremely hard to integrate, can we still say something about their convergence or divergence?

5.10 Comparison Theorem

Comparison Theorem. Suppose that f and g are continuous functions with $0 \le g(x) \le f(x)$ for $a \le x$. Then

$$0 \le \int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx$$

and we can conclude

1. If $\int_a^\infty f(x)\,dx$ **converges**, then $\int_a^\infty g(x)\,dx$ also **converges**.
2. If $\int_a^\infty g(x)\,dx$ **diverges**, then $\int_a^\infty f(x)\,dx$ also **diverges**.

Warning: The below two conclusions are **false**.

FALSE 1. If $\int_a^\infty f(x)\,dx$ **diverges**, then $\int_a^\infty g(x)\,dx$ also **diverges**.

FALSE 2. If $\int_a^\infty g(x)\,dx$ **converges**, then $\int_a^\infty f(x)\,dx$ also **converges**.

For every false statement, we can find a **counterexample** to show why it is false.

FALSE 1. If $\int_a^\infty f(x)\,dx$ **diverges**, then $\int_a^\infty g(x)\,dx$ also **diverges**.

Counterexample. Consider $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ with domain $[1, \infty)$. We have that $0 \leq g(x) \leq f(x)$. Note $\int_1^\infty \frac{1}{x}\,dx = \infty$ while $\int_1^\infty \frac{1}{x^2}\,dx = 1$ which shows that $\int_1^\infty f(x)\,dx$ diverges but $\int_1^\infty g(x)\,dx$ converges, giving us a countereexample to the false claim above.

FALSE 2. If $\int_a^\infty g(x)\,dx$ **converges**, then $\int_a^\infty f(x)\,dx$ also **converges**.

Counterexample. We can use the same example. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ with domain $[1, \infty)$. Then $\int_1^\infty g(x)\,dx$ converges but $\int_1^\infty f(x)\,dx$ diverges, which shows that the above claim is indeed false.

5.10 Comparison Theorem

Make sure to **check the hypotheses** to get credit on homework and exams.

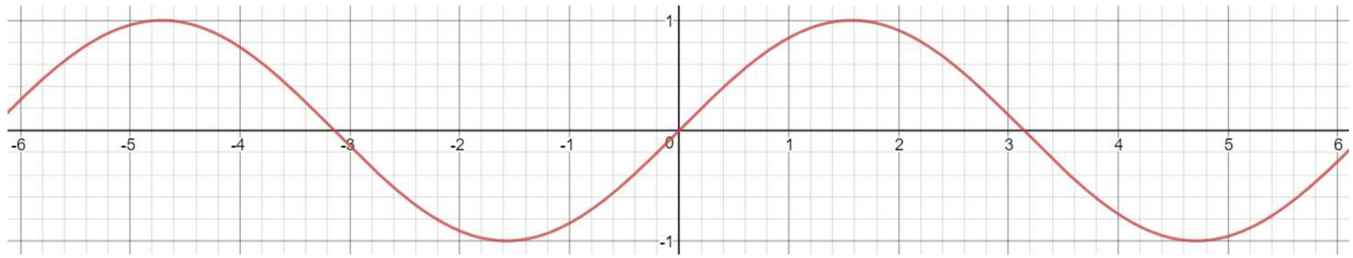
The hypotheses for the **Comparison Theorem**:

1. f and g are continuous.
2. $f(x) \geq g(x) \geq 0$ for $x \geq a$.

Trig inequalities to remember

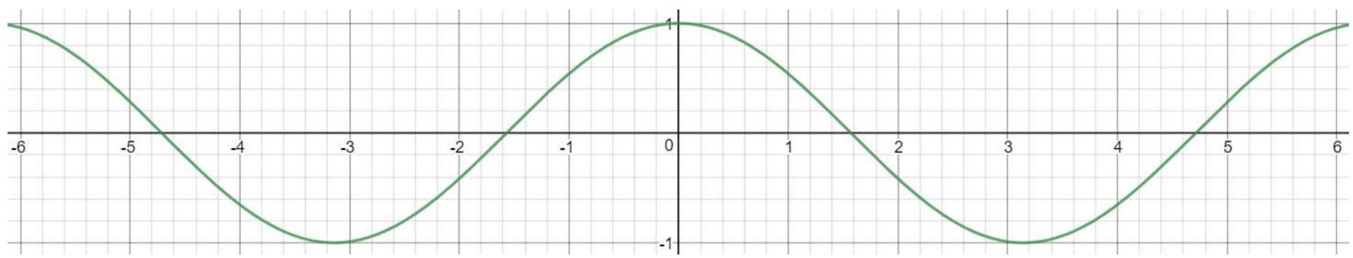
$$-1 \le \sin x \le 1$$

$$0 \le |\sin x| \le 1$$



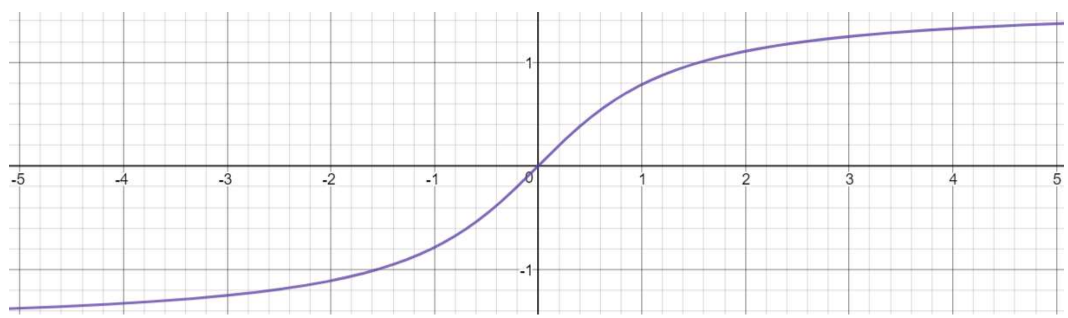
$$-1 \le \cos x \le 1$$

$$0 \le |\cos x| \le 1$$



$$\frac{-\pi}{2} < \arctan(x) < \frac{\pi}{2}$$

$$0 \le |\arctan(x)| < \frac{\pi}{2}$$



5.10 Improper Integrals (Comparison Test)

Determine whether $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx$ is convergent or divergent.

Choose a comparison function, check how it compares:

$0 \le |\sin x| \le 1$ for all x.

$$\frac{0}{x^2+1} \le \frac{|\sin x|}{x^2+1} \le \frac{1}{x^2+1} < \frac{1}{x^2}.$$

Check hypotheses for the comparison test:

$\dfrac{|\sin x|}{x^2+1}$ and $\dfrac{1}{x^2}$ are continuous and non-negative on $(1,\infty)$.

Therefore by the comparison test,

$$\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$$ and since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges by p-test $(p=2>1)$

$\displaystyle\int_1^\infty \frac{|\sin x|}{x^2}\,dx$ also converges.

5.10 Improper Integrals (Comparison Test)

Determine whether $\int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ is convergent or divergent.

Choose a comparison function, check how it compares.

$1 \leq \ln x$ for x in $(3,\infty)$

$0 < \frac{1}{\sqrt{x}} \leq \frac{\ln x}{\sqrt{x}}$ for x in $(3,\infty)$. use $\frac{1}{\sqrt{x}}$ as comparison.

Check hypotheses for the comparison test:

$\frac{1}{\sqrt{x}}$ and $\frac{\ln x}{\sqrt{x}}$ are continuous and non-negative on $(3,\infty)$.

Therefore by the comparison test,

$\int_3^\infty \frac{1}{\sqrt{x}}\,dx \leq \int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ and since $\int_3^\infty \frac{1}{\sqrt{x}}\,dx$ diverges by p-test ($p=\frac{1}{2}\leq 1$)

$\int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ also diverges.

5.10 Improper Integrals (Comparison Test)

Determine whether $\displaystyle\int_1^\infty \frac{1+e^x}{x}\,dx$ is convergent or divergent.

Choose a comparison function, check how it compares.

$1 \le 1+e^x$ for x in $(1,\infty)$

$0 < \frac{1}{x} \le \frac{1+e^x}{x}$ for x in $(1,\infty)$. Use $\frac{1}{x}$ as comparison.

Check hypotheses for the comparison test:

$\frac{1}{x}$ and $\frac{1+e^x}{x}$ are continuous and non-negative on $(1,\infty)$.

Therefore by the comparison test,

$\int_1^\infty \frac{1}{x}\,dx \le \int_1^\infty \frac{1+e^x}{x}\,dx$ and since $\int_1^\infty \frac{1}{x}\,dx$ diverges by the p-test $(p=1\le 1)$

$\int_1^\infty \frac{1+e^x}{x}\,dx$ diverges.

5.10 Improper Integrals (Comparison Test)

Determine whether $\displaystyle\int_3^\infty \frac{1}{x\ln x}\,dx$ is convergent or divergent.

Finding a comparable function can be tricky. $\dfrac{1}{x^2} \le \dfrac{1}{x\ln x} \le \dfrac{1}{x}$

but $\displaystyle\int_3^\infty \frac{1}{x^2}dx$ converges while $\displaystyle\int_3^\infty \frac{1}{x}dx$ diverges. convergent integral $\le \displaystyle\int_3^\infty \frac{1}{x\ln x}dx \le$ divergent integral

doesn't help us in trying to use the comparison test.

In this case, let's try to compute the integral.

$u = \ln x$

$du = \dfrac{dx}{x}$

$$\int_3^\infty \frac{1}{x\ln x}dx = \int_{\ln(3)}^{\ln(\infty)} \frac{du}{u} = \int_{\ln(3)}^{\infty} \frac{du}{u} = \lim_{t\to\infty}\int_{\ln(3)}^{t}\frac{du}{u} = \lim_{t\to\infty}\Big[\ln|u|\Big]_{\ln(3)}^{t}$$

Type 1 Improper

$$= \lim_{t\to\infty} \ln|t| - \ln|\ln(3)| = \infty - \ln|\ln(3)| = \infty.$$

Diverges to ∞

Therefore $\displaystyle\int_3^\infty \frac{1}{x\ln x}dx$ diverges

5.10 Improper Integrals (Comparison Test)

Determine whether $\int_0^\infty e^{-x^2}\, dx$ is convergent or divergent.

e^{-x^2} does not have an antiderivative. In this case, we break the integral into pieces and try to determine convergence for each piece.

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

Since e^{-x^2} is continuous and bounded over the interval $[0,1]$, $\int_0^1 e^{-x^2}dx$ is finite. You can check that $\int_0^1 e^{-x^2}dx$ is finite from the graph of e^{-x^2} and seeing that the area under the curve is finite.

For $\int_1^\infty e^{-x^2}dx$, we compare e^{-x^2} to e^{-x}. Observe that $e^{-x^2} = \frac{1}{e^{x^2}} < \frac{1}{e^x} = e^{-x}$ for $x > 1$ (you can check from the graph.) Since e^{-x^2} and e^{-x} are continuous and non-negative on $(1,\infty)$, we can use the comparison test: $\int_1^\infty e^{-x^2}dx \le \int_1^\infty e^{-x}dx$ and $\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = 0 + \frac{1}{e} < \infty$.

Since the bigger integral converges, the smaller integral must also converge by the comparison test. Therefore $\int_1^\infty e^{-x^2}dx$ converges.

Now to put it all together, we know that $\int_0^1 e^{-x^2}dx$ is finite (converges) and $\int_1^\infty e^{-x^2}dx$ converges (which means the integral is finite).

Since $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ is a sum of two finite integrals, $\int_0^\infty e^{-x^2}dx$ must be finite and so it is <u>convergent.</u>