Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

5.8 Tables of Integrals and Computer Algebra Systems

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don't have access to a computer algebra system (Wolfram Alpha, Mathematica, etc.).

Note that all of the formulas in the Tables of Integrals can be derived from techniques such as substitution, integration by parts, trig sub, and partial fractions.

We will explore how we can use Tables of Integrals to compute messy integrals.

5.8 Tables of Integrals

Use the Table of Integrals to find $\displaystyle\int \frac{x^2}{\sqrt{5-4x^2}}\,dx.$

The expression inside the square root resembles expressions of the form $\sqrt{a^2-u^2}$. We see that the closest entry is #34.

#34 $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right) + C$

The goal is to find the right u-substitution that transforms our original integral into the form prescribed by #34. $u = 2x$ will do the trick.

$u = 2x$
$du = 2\,dx$

$$\int \frac{x^2}{\sqrt{5-4x^2}}\,dx = \int \frac{(u/2)^2}{\sqrt{5-u^2}}\left(\frac{du}{2}\right) = \frac{1}{8}\int \frac{u^2}{\sqrt{5-u^2}}\,du$$

Now we use #34 with $a^2 = 5$ which means $a = \sqrt{5}$.

$$\frac{1}{8}\int \frac{u^2}{\sqrt{5-u^2}}\,du = \frac{1}{8}\left(-\frac{u}{2}\sqrt{5-u^2} + \frac{5}{2}\arcsin\left(\frac{u}{\sqrt{5}}\right)\right) + C = -\frac{x}{8}\sqrt{5-4x^2} + \frac{5}{16}\arcsin\left(\frac{2x}{\sqrt{5}}\right) + C$$

Use the Table of Integrals to find $\int x\sqrt{x^2+2x+4}\,dx$.

Looking at the Table of Integrals, we see that no formula resembles the integral above. To change its appearance, we first complete the square.

$x^2+2x+4=(x+1)^2+3$

If we make the substitution $u=x+1$, the integrand looks like $\sqrt{a^2+u^2}$.

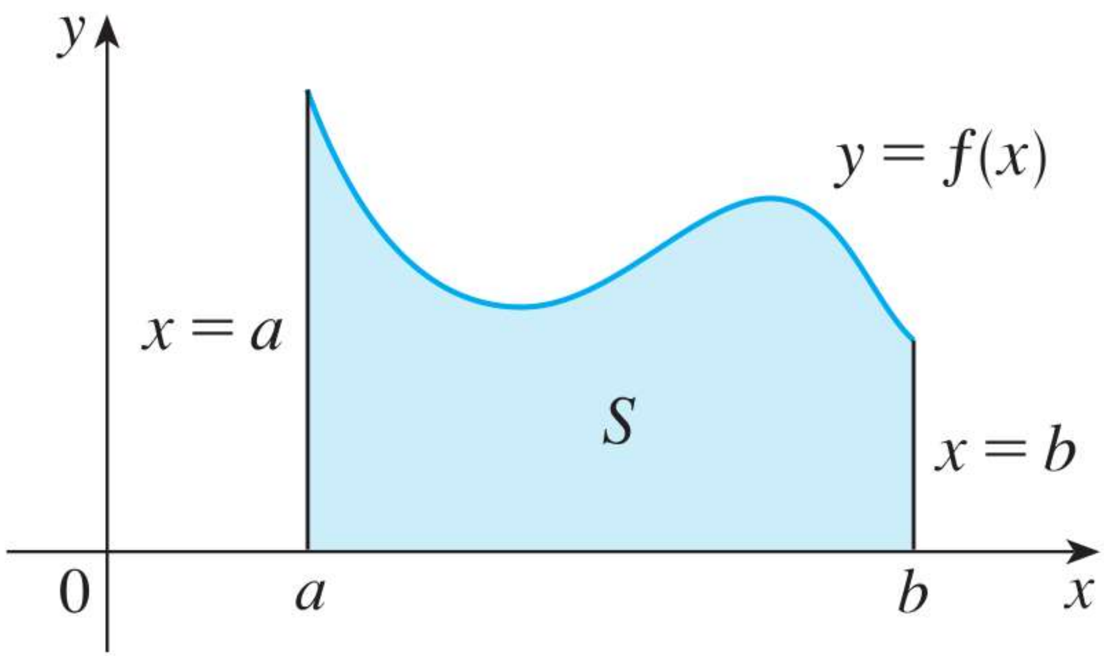
$$\int x\sqrt{x^2+2x+4}\,dx=\int (u-1)\sqrt{u^2+3}\,du=\underbrace{\int u\sqrt{u^2+3}\,du}_{\substack{\text{substitution}\\ t=u^2+3}}-\underbrace{\int\sqrt{u^2+3}\,du}_{\substack{\text{Integral Table \#21}\\ a=\sqrt{3}}}$$

$$=\frac{1}{3}(u^2+3)^{3/2}-\left[\frac{u}{2}\sqrt{u^2+3}+\frac{3}{2}\ln\left(u+\sqrt{u^2+3}\right)\right]$$

$$=\frac{1}{3}(x^2+2x+4)^{3/2}-\frac{(x+1)}{2}\sqrt{x^2+2x+4}-\frac{3}{2}\ln\left(x+1+\sqrt{x^2+2x+4}\right)+C$$

5.1 The Area Problem

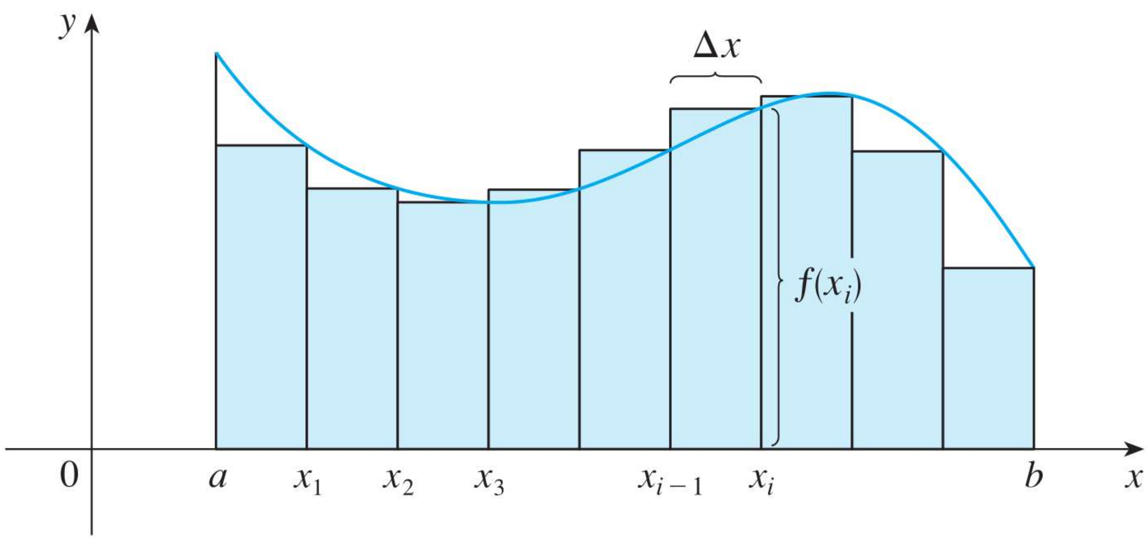
We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



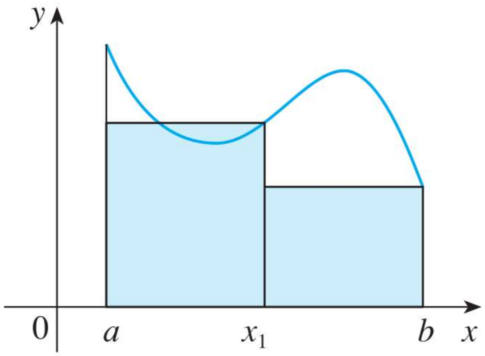
It isn't easy to find the area of a region with curved sides; however, we can **approximate** the area using rectangles. Let $x_1, x_2, x_3, \cdots$ denote the equally spaced x-coordinates along the base. Then the **height** of each rectangle is given by the function $f(x_i)$ while the **width** of each rectangle is given by the quantity $\Delta x = \dfrac{b-a}{n}$. This means each rectangle's area is **height·width**$= f(x_i)\ \Delta x$. If we have n rectangles, then the area under the curve is approximately

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

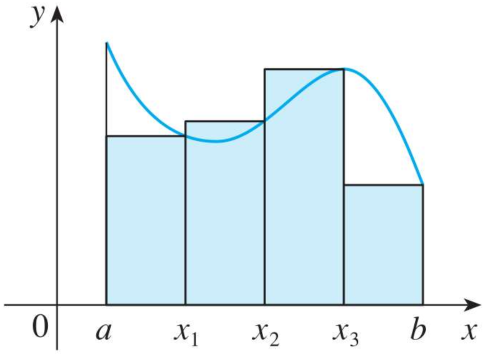
We call the above sum a **Riemann sum**.



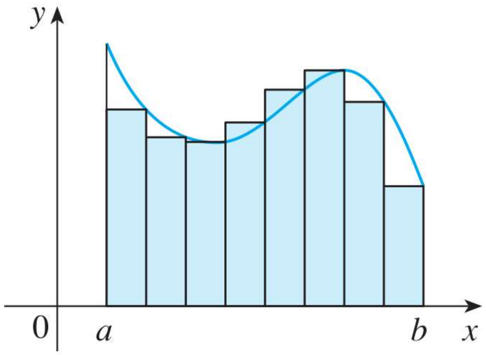
Notice that this approximation appears to become better and better as the number of rectangles increases, that is, as $n \to \infty$.



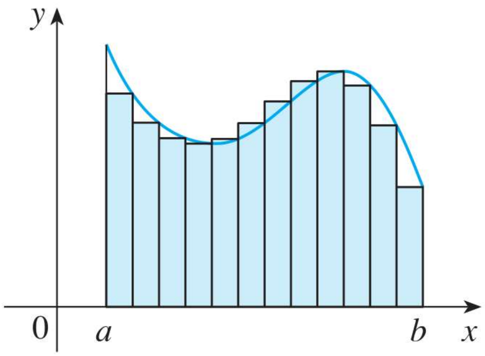
(a) $n = 2$



(b) $n = 4$



(c) $n = 8$



(d) $n = 12$

Therefore we define the **area** A under the graph of the function f as the limit of the sum of the areas of rectangles as we take the number of rectangles to infinity:

$$A = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

This gives us the definition of the definite integral:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

given that $\Delta x = \dfrac{b-a}{n}$.

5.9 Approximate Integration

When do we need to approximate integrals?

- Finding the exact value can be costly – time, energy.
- Not every function has an antiderivative.
- For example, consider the integral below:

$$\int_0^1 e^{-x^2}\ dx$$

The function e^{-x^2} has no antiderivative but we can still approximate the area under the curve using Riemann sums.

5.9 Approximate Integration

How do we approximate integrals?

- To simplify things, we are often given either a formula or a table of values.
- For example, the data collected from instrument readings during scientific experiments gives us a table of values.
- We use Riemann sums to approximate our integral.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\triangle x = f(x_1)\triangle x + f(x_2)\triangle x + \cdots + f(x_n)\triangle x$$

5 Types of Riemann Sums

Let n denote the number of rectangles used in the approximation. For each method, the rectangles are evenly spaced. $x_0 = a$, $x_n = b$, and $\Delta x = \dfrac{b-a}{n}$.

1. Left endpoint: L_n uses the **left-most** x-coordinate of each subinterval to find the height of each rectangle.

2. Right endpoint: R_n uses the **right-most** x-coordinate of each subinterval to find the height of each rectangle.

3. Midpoint: M_n uses the **middle** x-coordinate of each subinterval to find the height of each rectangle.

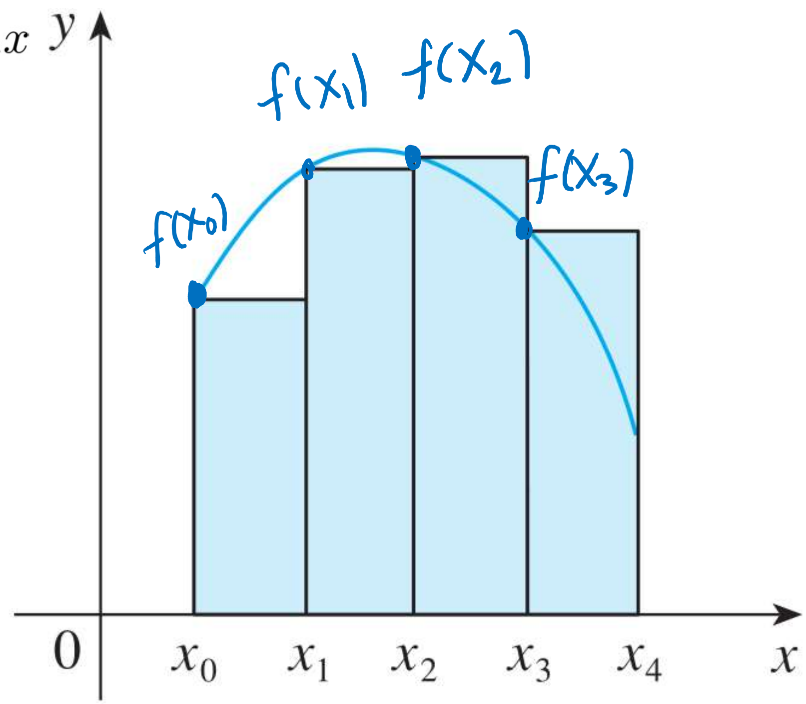
4. Trapezoidal (average height): T_n uses the **average** of the heights from the Left endpoint and the Right endpoint to find the height of each rectangle.

5. Simpson's Rule (parabolic): S_n uses sections of parabolas to estimate the areas.

1. Left endpoint Approximation

$$\int_a^b f(x)dx \approx L_4 = f(x_0)\triangle x + f(x_1)\triangle x + f(x_2)\triangle x + f(x_3)\triangle x$$

Choose the **left-most** x-coordinate in each subinterval to sample the height.

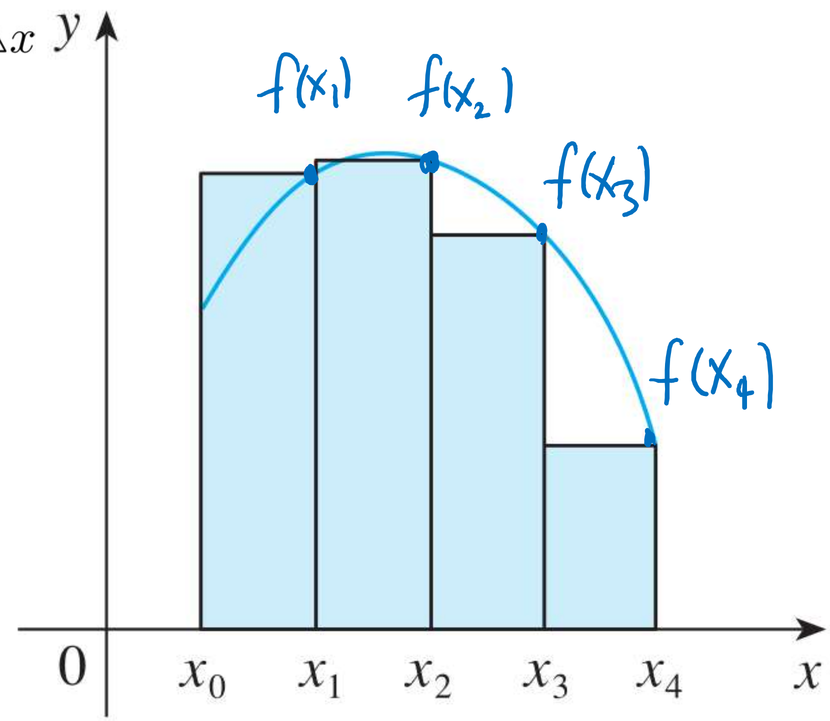


(a) Left endpoint approximation

2. Right endpoint Approximation

$$\int_a^b f(x)dx \approx R_4 = f(x_1)\triangle x + f(x_2)\triangle x + f(x_3)\triangle x + f(x_4)\triangle x$$

Choose the **right-most** x-coordinate in each subinterval to sample the height.



(b) Right endpoint approximation

3. Midpoint Approximation

$$\int_a^b f(x)dx \approx M_4 = f(\overline{x}_1)\triangle x + f(\overline{x}_2)\triangle x + f(\overline{x}_3)\triangle x + f(\overline{x}_4)\triangle x$$

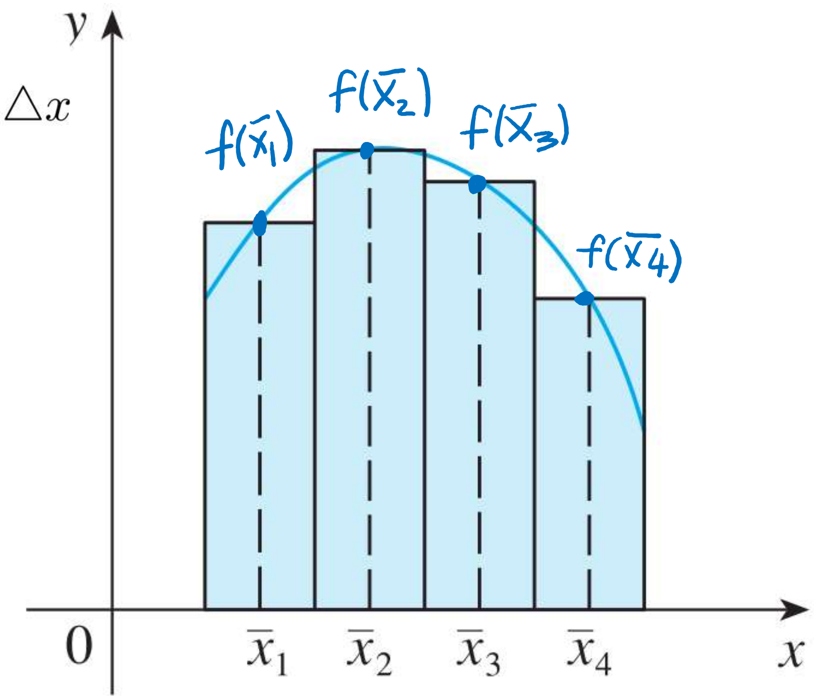
Choose the **middle** x-coordinate in each subinterval to sample the height.

$$\overline{x_1} = \frac{x_0 + x_1}{2}$$

$$\overline{x_2} = \frac{x_1 + x_2}{2}$$

$$\overline{x_3} = \frac{x_2 + x_3}{2}$$

$$\overline{x_4} = \frac{x_3 + x_4}{2}$$



(c) Midpoint approximation

4. Trapezoidal Approximation – Average height

$$\int_a^b f(x)dx \approx T_4 = \frac{\triangle x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

Average the heights from the left endpoint and the right endpoint.

$$h_1 = \frac{f(x_0) + f(x_1)}{2}$$

$$h_2 = \frac{f(x_1) + f(x_2)}{2}$$

$$h_3 = \frac{f(x_2) + f(x_3)}{2}$$

$$h_4 = \frac{f(x_3) + f(x_4)}{2}$$

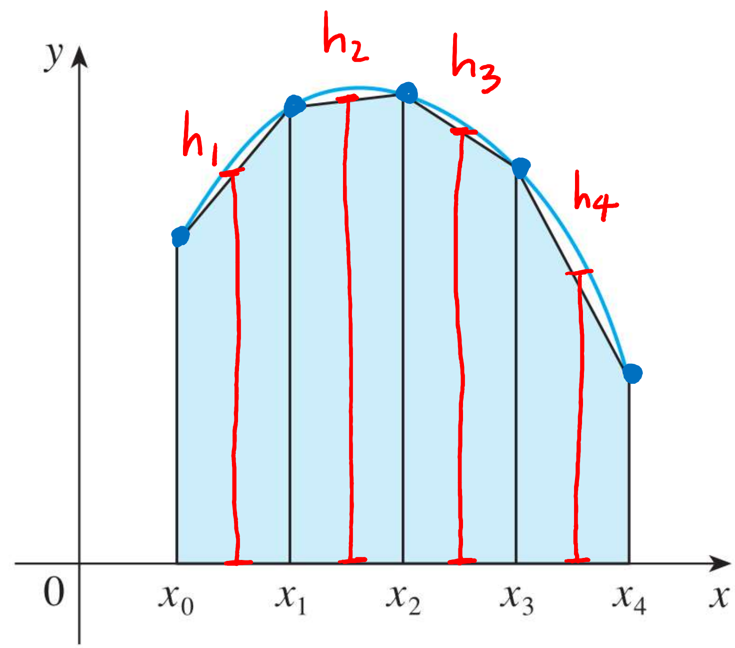


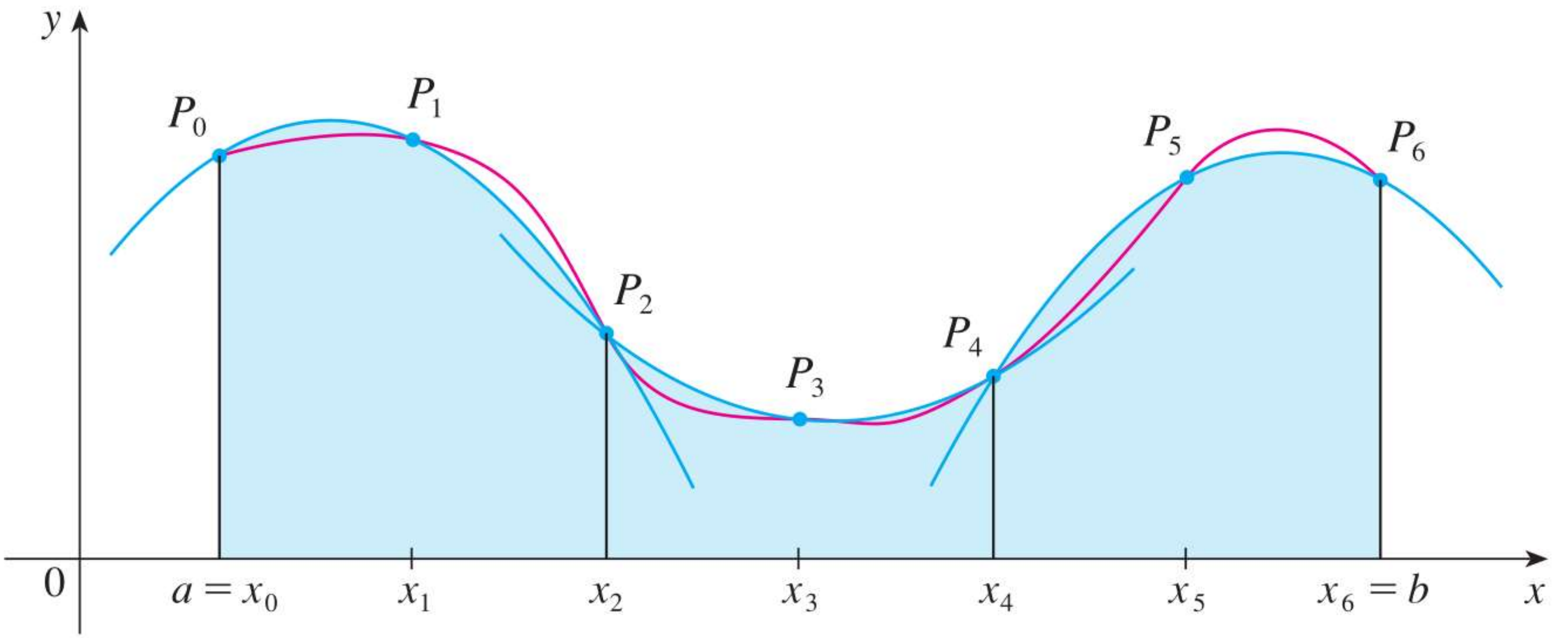
FIGURE 2

Trapezoidal approximation

5. Simpson's Rule – Sections of Parabolas

For Simpson's Rule, we create **one** parabolic rectangle using **two** consecutive subintervals; in other words, if there are n subintervals, then we have $\frac{n}{2}$ rectangles. Therefore n must be **even**; otherwise, we can't use Simpson's Rule.

$$\int_a^b f(x)dx \approx S_6 = \frac{\triangle x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)]$$



5.9 Approximate Integration

- Left endpoint, right endpoint, midpoint, trapezoidal methods, and Simpson's Rule are all used to approximate integrals.
- Since these are approximations, they are off by some error from the true value.
- How do we control the error to guarantee accuracy?

5.9 Approximate Integration

3 **Error Bounds** Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

Example 1: If we use the trapezoidal approximation with $n = 10$ to estimate $\int_1^3 x^3\,dx$, how accurate are we guaranteed to be? (If you want, make a guess before you do the calculation.)

$f(x) = x^3$

$f'(x) =$ $3x^2$

$f''(x) =$ $6x$

On $[1, 3]$, $|f''(x)| \leq$ $18 = K$, because f'' is increasing on the interval $[1,3]$ so we plug in the right-most x-value.

So, $|E_T| \leq \dfrac{K(b-a)^3}{12n^2} = \dfrac{18(3-1)^3}{12(10)^2}$ (Is this more or less accurate than you guessed?)

$= \dfrac{18 \cdot 8}{1200} = \dfrac{144}{1200} = 0.12$

Example 2: If we use the midpoint approximation with $n = 20$ to estimate $\int_0^1 \sin(2x)\,dx$, how accurate are we guaranteed to be?

$f(x) = \sin(2x)$

$[a,b] = [0,1]$

$|E_M| \leq \frac{K(b-a)^3}{24n^2}$

$|f''(x)| \leq K$ for $a \leq x \leq b$

$f''(x) = -4\sin(2x)$

$|-4\sin(2x)| \leq 4 = K$

$$|E_M| \leq \frac{4(1-0)^3}{24(20)^2} = \frac{1}{6 \cdot 400} = \frac{1}{2400} = 4.17 \times 10^{-4}$$

Example 3: How large should n be to guarantee that using T_n to estimate $\int_0^1 e^{-3x}\,dx$ gives an error no larger than 0.001?

$|E_T| \le \frac{K(b-a)^3}{12n^2}$ $\quad |f''(x)| \le K$ for $a \le x \le b$

$f(x) = e^{-3x}$ $\quad [a,b] = [0,1]$

$f''(x) = 9e^{-3x}$

$9e^{-3x}$ is a decreasing function so it is maximized at $x=0$ for x in $[0,1]$.

$|9e^{-3x}| \le 9e^0 = 9 = K$

$|E_T| \le \frac{9(1-0)^3}{12n^2} \le 0.001$

$\frac{9}{12n^2} \le 0.001$

$\frac{9}{12(0.001)} \le n^2$

$n \ge 27.3861$

$n \ge 28$ since n is the number of rectangles which needs to be a whole number.

5.9 Approximate Integration

Simpson's Rule (Using parabolas to approximate)

4 Error Bound for Simpson's Rule Suppose that $|f^{(4)}(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}$$

Example 4: How large should n be to guarantee that using the Simpson's Rule S_n to estimate $\int_0^1 e^{-3x}\,dx$ gives an error no larger than 0.001?

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

$|f^{(4)}(x)| \le K$ for $a \le x \le b$

$[a,b] = [0,1]$

$f(x) = e^{-3x}$

$f^{(4)}(x) = (-3)^4 e^{-3x}$

$f^{(4)}(x) = 81 e^{-3x}$

$|f^{(4)}(x)| \le 81 e^0 = 81 = K$

$$|E_S| \le \frac{81(1-0)^5}{180n^4} \le 0.001$$

$$\frac{81}{180n^4} \le 0.001$$

$$\frac{81}{180(0.001)} \le n^4$$

$n \ge 4.60578$

$n \ge 6$ since Simpson's Rule requires n to be even.

since $81e^{-3x}$ is a decreasing function and is maximized when $x=0$ in the interval $[0,1]$.