Quiz Scores

- Lecture attendance gets extra credit
- 4 lectures = 1 extra quiz point

Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

5.7 Partial Fractions (Repeated Factors)

Find $\displaystyle\int \frac{x+3}{(x-1)^2}\,dx$.

Note: For repeated factors like "$(x-1)^2$" in the denominator, the Residue method doesn't work because it assumes each factor has power 1.

The desired form of partial fractions for a twice-repeated factor is

$$\frac{x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

We use the 5-step process to find A and B to solve the integral.

① Clear the denominator

$$\frac{x+3}{(x-1)^2}(x-1)^2 = \frac{A}{x-1}(x-1)^2 + \frac{B}{(x-1)^2}(x-1)^2$$

$$x+3 = A(x-1) + B$$

5.7 Partial Fractions (Repeated Factors)

Find $\int \frac{x+3}{(x-1)^2}\,dx$.

② Collect the coefficients

$$x+3 = x[A] + [-A+B]$$

③ Set the coefficients equal

$$1 = A$$
$$3 = -A + B$$

④ Solve for A and B

$$A = 1$$
$$3 = -1 + B$$
$$4 = B$$

⑤ Integrate

$$\int \frac{x+3}{(x-1)^2}\,dx = \underline{\int \frac{1}{x-1}\,dx} + \underline{\int \frac{4}{(x-1)^2}\,dx}$$

Simple u-sub

$$= \boxed{\ln|x-1| + 4\left(-(x-1)^{-1}\right) + C}$$

5.7 Partial Fractions (Irreducible Factors)

Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx.$

A non-constant polynomial is **irreducible** if its coefficients are real numbers and it cannot be factored into a product of **two** non-constant polynomials with real coefficients. For real polynomials, irreducible polynomials are either degree 1 or degree 2.

Let's factor the denominator.

$$x^3 + 4x = x(x^2+4) = x(x+2i)(x-2i)$$

Observe that x^2+4 cannot be factored over the real numbers because it has only complex roots. Hence x^2+4 is irreducible.

With irreducible polynomials in the denominator, we use a different technique for partial fraction decomposition.

5.7 Partial Fractions (Irreducible Factors)

Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx.$

The desired form is

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{\overbrace{Bx+C}}{x^2+4}$$

The numerator's degree for the set-up is 1 fewer than the degree of the denominator as long as the denominator is <u>not</u> a repeated factor.

The Residue Method doesn't work in this case so we use the 5-step method.

① Clear the denominator

$$\frac{2x^2 - x + 4}{x(x^2+4)}\, x(x^2+4) = \frac{A}{x}\, x(x^2+4) + \frac{Bx+C}{x^2+4}\, x(x^2+4)$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)\,x$$
$$= Ax^2 + 4A + Bx^2 + Cx$$

5.7 Partial Fractions (Irreducible Factors)

Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx.$

② Collect the coefficients

$$2x^2 - x + 4 = x^2[A+B] + x[C] + [4A]$$

③ Set the coefficients equal

$$2 = A+B$$
$$-1 = C$$
$$4 = 4A$$

④ Solve for A, B, C

$$C = -1$$
$$A = 1$$
$$B = 1$$

⑤ Integrate

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\, dx = \int \frac{1}{x}\, dx + \int \frac{x-1}{x^2+4}\, dx$$

$$= \ln|x| + \int \frac{x}{\underbrace{x^2+4}_{\substack{u\text{-sub} \\ u = x^2+4}}}\, dx - \int \frac{1}{\underbrace{x^2+4}_{\substack{u\text{-sub} \\ u = \frac{x}{2}}}}\, dx$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

5.7 Partial Fractions (Higher degree in the numerator)

Use long division to evaluate $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$.

If the degree of the numerator is higher than that of the denominator, we use long division and perform partial fraction decomposition on the remainder.

$$
\begin{array}{r|l}
 & \quad x \\ \hline
x^2+4 & x^3 + 0x^2 + 0x + 4 \\
 & -(x^3 + \qquad 4x) \\ \hline
 & 0 + 0 - 4x + 4
\end{array}
\qquad R = -4x+4
$$

Hence $$\frac{x^3+4}{x^2+4} = x + \frac{-4x+4}{x^2+4}$$

5.7 Partial Fractions (Higher degree in the numerator)

Use long division to evaluate $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$.

Now we can use partial fraction decomposition on $\dfrac{-4x+4}{x^2+4}$. But observe that x^2+4 is irreducible. Hence no further work is needed with decomposition. We can then integrate:

$$\int \frac{x^3+4}{x^2+4}dx = \int x + \frac{-4x+4}{x^2+4}\,dx$$

$$= \int x\,dx - \int \frac{4x}{x^2+4}dx + \int \frac{4}{x^2+4}dx$$

$$= \boxed{\frac{x^2}{2} - 2\ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right) + C}$$