Exam 2 Review Handout

1. Sequences

A **sequence** $\{a_n\}$ is a list of numbers written in a definite order:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

Give an example or two:

Convergence and **divergence** of mathematical objects like sequences and series is about whether the limit exists or doesn't exist.

2. The Geometric Sequence

The **sequence** $a_n = r^n$ is **convergent** if $-1 < r \leq 1$ and **divergent** for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Give an example or two:

3. The Squeeze Theorem

If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Give an example or two:

1. Series

Given a sequence $\{a_n\}$, a **finite sum**

$$s_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + \cdots + a_m$$

is called the m-th **partial sum** s_m.

A **series** is an **infinite sum** of the sequence a_n, where

$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=1}^{m} a_n = \lim_{m\to\infty} s_m = s$$

If the above limit exists, we say that the series **converges** and if the above limit doesn't exist, then we say that the series **diverges**.

Give an example or two:

A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Give an example or two:

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is not absolutely convergent but still converges.

Give an example or two:

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Give an example or two:

2. Geometric Series

The geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

Give an example or two:

3. Telescoping Sums

With some algebra, a series can be broken down into a sum of a difference

$$\sum_{n=1}^{\infty} (a_n - a_{n+1})$$

where cancellation happens in the partial sum

$$\sum_{n=1}^{m} a_n - a_{n+1} = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_m - a_{m+1}) = a_1 - a_{m+1}$$

Take the limit of the partial sums as $m \to \infty$ to determine convergence.

Give an example or two:

4. The p-series and the p-test

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Give an example or two:

5. **Divergence Test**

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Give an example or two:

6. Convergent series must have vanishing terms at infinity.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Give an example or two:

7. **Integral Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and let $f(n) = a_n$. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$.

(a) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ diverges.

Give an example or two:

8. **Direct Comparison Test.**

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $0 \leq a_n \leq b_n$ for all n. Then

$$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

and

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

To use either of the comparison tests, we need to compare our messy-looking series to another series that we already understand. Below are the series that we understand so far:

(a) A geometric series (a and r are constants)

$$\sum_{n=0}^{\infty} ar^n$$

(b) A p-series (p is a constant)

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

(c) A series that looks similar to an improper integral that can be solved using u-sub or other integration techniques

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \approx \int_2^{\infty} \frac{1}{x \ln x}\, dx$$

Give an example or two:

9. **Limit Comparison Test**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is non-zero, then either both series converge or both series diverge.

Give an example or two:

10. **Alternating Series Test**

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}\, b_n$ is an alternating series. If

(a) $\lim_{n\to\infty} b_n = 0$ (vanishing at infinity)

(b) $b_n \geq b_{n+1}$ (decreasing)

then the alternating series is convergent.

Give an example or two:

11. **Ratio Test** (Use this if you see a factorial in the sum)

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

(a) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If $L = 1$, then the Ratio Test is inconclusive and we must use other testing methods.

Give an example or two:

8.3 Remainder Estimate for the Integral Test.

Suppose $f(k) = a_k$, where $f(x)$ is a continuous, positive decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$ where s_n is the n-th partial sum, then

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx.$$

Also,

$$s_n + \int_{n+1}^{\infty} f(x)\ dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\ dx.$$

Give an example or two:

8.4 Alternating Series Estimation Theorem.

If $\sum_{n=1}^{\infty}(-1)^{n-1}\ b_n = s$ is the sum of an alternating series that satisfies

$$\text{(i)}\ \lim_{k\to\infty} b_k = 0 \qquad \text{and} \qquad \text{(ii)}\ b_k \geq b_{k+1}$$

then $|R_n|$, the error for the n-th partial sum, is less than or equal to the $(n+1)$-th term, b_{n+1}.

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Note that $s_n = \sum_{k=1}^{n}(-1)^{k-1}b_k$. In other words, the error will be less than or equal to the next term.

Give an example or two: